Anyonic statistics of quasiparticles in the quantum Hall effect

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Master thesis

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May 2010

Acknowledgements

I would like to thank my supervisor Susanne Viefers for introducing me to the versatile topic of quantum Hall physic and providing plenty of help along the way. In addition the following people have been very helpful: Jon Magne Leinaas, T. Hans Hansson, Jaakko Nissinen, Juha Suorsa and Mats Horsdal. I also thank you friends, family and especially you Silje, for all the rest besides physics.

Oslo, May 26. 2010
Jørgen Fulsebakke

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Introduction

1.1 Preface

The quantum Hall effect (QHE) is a phenomenon which has created much interest since its discovery in 1980. It has been associated with exotic properties such as two-dimensionality, interactions that cannot be described perturbatively, fractional charge and statistics angles, resistance not depending on sample properties, non-abelian statistics and prospects of quantum computing.

The subject of this thesis is fractional statistics. This refers to particles whose wavefunction attains a fractional phase θ after interchange, called the statistics angle, rather than the ordinary 0 and π of bosons and fermions respectively. Particles with such a generalized angle are generically known as anyons and are predicted by different models to exist in the quantum Hall effect, although no universally accepted experimental observation of such particles exist yet. Even though the different theories explaining the effect have fermionic electrons as the basic particles, anyonic *quasi*particles emerge from all of them.

Our main objective with this text is to look at different models describing the QHE and what they say about the statistics of the quasiparticles. Since the various approaches use fairly different reasoning it is not given that they agree, and discrepancies could potentially discriminate between them. After a short introduction to the field and some basic theory we will present each model one at a time and go into them with as much depth as seems pertinent with our goal in mind. In the last chapter we generalize some older arguments to accomodate newer theoretical insights before summarizing what we have found and comparing the results concerning the statistics angle.

1.2 The Classical Hall Effect

The classical Hall effect, predicted by Edwin Hall in 1879 [1], occurs when a thin conducting slab in a perpendicular magnetic field has a current passing through. We will assume that the magnetic field is pointing along the negative z-axis throughout. As the current density $\mathbf{j}$ flows through the plate the magnetic Lorentz force $\mathbf{F}_L = Q\mathbf{v} \times \mathbf{B}$ causes the charge carriers of charge Q to accelerate perpendicularly both to the current and to the magnetic field $\mathbf{B}$ so that a surplus of charge builds up on one edge and a deficit on the opposite. The resulting electric field accumulates until its effect on the carriers matches the effect of the Lorentz force; at which point there is an equilibrium with $\mathbf{j}$ traveling straight through the sample and a Hall

voltage V_H normal to this. The effect is sensitive to charge density and carrier charge and can be used to measure both.

Using relativistic arguments and a semiclassical model we can derive the resistivity of an ideal plate. If we assume that we are in a frame A_0 with no external electric field but a magnetic field along the negative z-axis, we have

$$\mathbf{B}_0 = -B\mathbf{e}_z \qquad \mathbf{E}_0 = 0 \tag{1.2.1}$$

Where $\mathbf{e}_i$ is a basis vector in the i-direction and $B > 0$. Transforming to a second frame A (which is the frame where the Hall effect takes place) moving with a relative velocity $-\mathbf{v}$ to A_0 in a direction perpendicular $\mathbf{e}_z$ we see a current density

$$\mathbf{j} = \rho Q\mathbf{v} \tag{1.2.2}$$

Where ρ is the carrier density. Using Lorentz transformations we find that the fields in this frame are

$$\mathbf{B} = -B\mathbf{e}_z \qquad \mathbf{E} = -\mathbf{v} \times \mathbf{B}_0 = \frac{B}{\rho Q}\mathbf{j} \times \mathbf{e}_z \tag{1.2.3}$$

In tensor notation $E^\mu = \rho_{\mu\nu} j^\nu$ this gives us for the resistivity tensor

$$[\rho_{\mu\nu}] = \frac{B}{\rho Q}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.2.4}$$

Thus the longitudinal components are zero while the transverse, referred to as the Hall resistivity, are proportional to the ratio of magnetic field to carrier density. In a real sample disorder and impurities, not taken into account in the above argumentation, will produce a dissipative resistivity but not change the transverse components.

1.3 The Quantum Hall Effect

The 1980 discovery of the Quantum Hall Effect (QHE) [2] was contradictory to the above result. It was found that when the system is effectively two-dimensional, the sample is very pure (that is, has little disorder and a high mobility), the magnetic field is strong and the temperature approaches zero the resistivity was measured to[1]

$$[\rho_{\mu\nu}] = \frac{1}{\nu}\frac{h}{e^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.3.1}$$

where h is Planck's constant and $-e$ is the fundamental electron charge[2]. But instead of being linearly proportional to B the transverse resistivity was constant on plateaus corresponding to integer ν. The longitudinal resistivity was found to disappear at these same values of B, with maxima at the transitions between plateaus. In addition the effect was seen to be universal, with little dependence on the properties of the sample.

With even purer samples the effect was in 1983 discovered also with fractional ν [3], prompting a division into the integer (IQHE) and fractional (FQHE) effects. Figure 1.1 displays these results.

[1] Of course in an experiment it is the resistance rather than the resistivity that is measured. But in the QHE the two are equal so that the resistance is unaffectet by sample geometry, enabling extremely precise measurements. Due to dimensional considerations this is only possible in 2D.

[2] We will use this convention for e throughout, implying $e = |e|$.

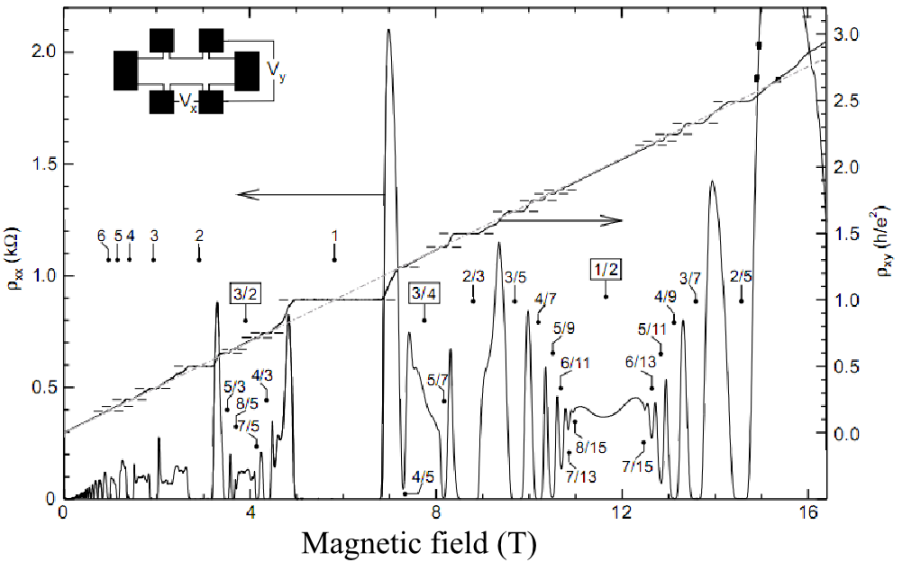

Figure 1.1: The quantum Hall effect: Transversal (right side) and longitudinal (left side) resistivity vs. magnetic field. Measured by J. Smeta at MPI-Institut Stuttgart, Germany. Taken with permission from [7].

1.4 Quasiparticles

There are different models explaining the QHE, and we will examine several of these. But they all feature something called quasiparticles (qp). These are local, fractionally charged entities other than electrons - even though the theory is fundamentally treating electrons in a magnetic field exclusively. This means that the quasiparticles must be an emergent effect in the system, where the properties of the electrons (interactions and symmetry constraints) result in a homogeneous electron gas with localized charge deviations.

From one standpoint these qp's are composite objects less fundamental than the underlying electron system. But from another point of view it depends on what level of description we choose our basic system - if the QHE state is taken as the vacuum then the qp's are the elementary dynamic objects. This is in some ways similar to how one may describe protons and neutrons as fundamental entities with Yukawa potentials or on another level as composed of quarks described by quantum chromodynamics.

Reference [4] illustrates the comparison between high energy physics' search for new phenomena and QHE quasiparticles with the following analogy: Imagine flatlanders living in a two dimensionional low temperature world - a quantum Hall world. Their elementary particles are the QHE qp's, and if they want to study our elementary particles they must reach a high energy and thus excite an electron out of the vacuum.

In addition the theory predicts that these quasiparticles obey anyonic statistics, that is they are neither of the familiar (in three dimensions) fermions or bosons. Our aim is to compare what the different models have to say about this.

1.5 Topology

The mathematics of topology can be described [5] as the study of certain properties that are preserved by continuous transformations[3] of objects called topological spaces. If two spaces can be transformed into each other continuously they are said to be homeomorphic; this is the most basic form of topological equivalence. A cube and a square are homeomorphic, as are doughnut and teacup shapes, while a cube and a doughnut or a circle and a doughnut are not. Other examples of topological concepts are compactness, connectivity, homotopy groups and manifolds. Beginning as investigations in geometry, topology is now a collection of abstract tools based on set theory that are useful in many different physical situations: Topological spaces arise naturally in many mathematical contexts and topology may be used to characterize these spaces and handle in an efficient way questions that do not depend on the exact shape of the objects involved but more general properties.

The main topic of this thesis concerns anyons. In showing that these particles can exist we will examine the topology of configuration spaces, in particular the difference between these in three and two spatial dimensions.

Topological quantum numbers are important because they are robust to changes in the system, e.g. alterations of the Hamiltonian or the wavefunction (as opposed to for example classical angular momentum, which is conserved when we have rotational symmetry but not if there is a slight deviation from this). The Hall resistivity can be thought of as such a number [6], and this explains its indifference to the exact geometry of the sample. As we will see the statistical angles of anyons depend on the topological structure of the particle configuration space and are therefore also topological quantum numbers. It should however be mentioned that there is no strict definition as to what constitutes such a number.

[3] For example deformations including stretching but no tearing or gluing.

Chapter 2

Basic theory of the QHE

Following is a summary of the relevant basic theory, based on [4, 7–10]

2.1 Charged particle in a magnetic field

The basic ingredient of the QHE models is a single electron confined to two dimensions in a perpendicular magnetic field (in our case along $-\mathbf{e}_z$). In an actual experiment the relevant electrons reside between semiconductors so as to effectively trap them in two dimensions; a fully rigorous treatment would have to include more than just the electrons. But we will assume that apart from the periodical background ion lattice leading to an effective band mass of the electrons instead of the usual rest mass these effects can be ignored. The agreement between the models and measurements show that this approach is justified. In addition to the theory we will see in this section we shall later need two more basic ingredients; disorder and interactions. For the entirety of this text we will assume full polarization and therefore neglect spin.

We introduce a Cartesian coordinate system and use as independent coordinates $z \equiv x+iy$ and $\overline{z} \equiv x - iy$. The classical equation of motion from the magnetic Lorentz force is[1]

$$m\ddot{\mathbf{r}} = -e\dot{\mathbf{r}} \times \mathbf{B} \qquad \Longrightarrow \quad \ddot{z} = -i\omega_c \dot{z} \tag{2.1.1}$$

where we define $\omega_c \equiv \frac{eB}{m}$. The solution to the above differential equation may be integrated:

$$\dot{z} = Ae^{-i\omega_c t} \qquad \Longrightarrow \quad z = C + \frac{i}{\omega_c} Ae^{-i\omega_c t} \tag{2.1.2}$$

where C is an integration constant. The result is cyclotron motion with frequency ω_c around the guiding center C, which is a constant of motion.

Turning to the quantum mechanical version we use minimal substitution and find the single-particle Hamiltonian

$$H = \frac{1}{2m}\mathbf{\Pi}^2 \quad \text{where} \quad \mathbf{\Pi} \equiv \mathbf{p} + e\mathbf{A}(\mathbf{r}) \tag{2.1.3}$$

$\mathbf{p}$ is mechanical momentum and $\mathbf{A}$ is the electromagnetic vector potential. All vectors are two-dimensional. The components of the gauge invariant momentum do not commute:

$$[\Pi_x, \Pi_y] = [-i\hbar\partial_x, eA_y] + [-i\hbar\partial_y, eA_x] = -ie\hbar(\nabla \times \mathbf{A}) \cdot \mathbf{e}_z = \frac{i\hbar^2}{l^2} \tag{2.1.4}$$

[1] We use units where $c = 1$ throughout.

where $l^2 \equiv \frac{\hbar}{eB}$ and we have used that $\mathbf{A} \perp \mathbf{e}_z$.

For now we use a gauge-independent, algebraic treatment and to this end introduce the operators

$$\alpha \equiv \frac{l}{\sqrt{2}\hbar}(\Pi_x + i\Pi_y) \qquad \alpha^\dagger \equiv \frac{l}{\sqrt{2}\hbar}(\Pi_x - i\Pi_y)$$
$$\implies \quad [\alpha, \alpha^\dagger] = 1 \qquad H = \hbar\omega_c(\alpha^\dagger\alpha + 1/2) \tag{2.1.5}$$

We recognize the harmonic oscillator algebra and thus obtain the discrete energy levels $E_n = \omega_c(n + 1/2)$ where $n \in \mathbb{N}$. In this context these are called Landau levels (LL) and there is a lowest Landau level (LLL) such that $\alpha|LLL\rangle = 0$ and $H|LLL\rangle = E_0|LLL\rangle = \omega_c/2|LLL\rangle$.

Motivated by the guiding center in the classical description we examine the operator $C \equiv z - \frac{i}{m\omega_c}\Pi$ where $\Pi \equiv \Pi_x + i\Pi_y$ (guided by (2.1.2) and $\dot{z} \to \Pi/m$). The components of C are also canonical conjugates, which means that we have another independent harmonic oscillator algebra:

$$[C_x, C_y] = -il^2 \qquad \beta \equiv \frac{1}{\sqrt{2}l}(C_x + iC_y) \qquad \beta^\dagger \equiv \frac{1}{\sqrt{2}l}(C_x - iC_y)$$
$$\implies \qquad [\beta, \beta^\dagger] = 1 \qquad [\alpha, \beta] = [\alpha, \beta^\dagger] = [H, \beta] = [H, \beta^\dagger] = 0 \tag{2.1.6}$$

Note that in this case we cannot use the argument $E > min(V)$ to demand that there is a state for which $\beta|\psi\rangle = 0$ as is usually done with the harmonic oscillator [11]. But we see that the operator $\beta^\dagger\beta$ must have non-negative eigenvalues to adher to the probability postulate because $\langle\psi_m|\beta^\dagger\beta|\psi_m\rangle \sim \langle\psi_{m-1}|\psi_{m-1}\rangle \geq 0$. Since β lowers the eigenvalue of $\beta^\dagger\beta$ by one unit there must be a state $|0\rangle$ that is annihilated by β, to avoid negative eigenvalues

The fact that β commutes with the Hamiltonian means that we may use this operator to obtain different states with the same energy, highlighting the degeneracy of the LL's. Because C can be expressed in β and $\beta^\dagger$ it also means that C is a constant of motion so that we retain a residue of the classical cyclotron motion (as shown semiclassically in [4] small wavepackets will behave like the classical particles). Since α and β commute we have the complete set of states $|n\ m\rangle \sim (\alpha^\dagger)^n(\beta^\dagger)^m|0\ 0\rangle$. We see that n and m are the eigenvalues of $\alpha^\dagger\alpha$ and $\beta^\dagger\beta$ respectively and use these as quantum numbers to label our single-particle states.

At this point we introduce the symmetric gauge, which we will use often:

$$\mathbf{A}_{sym} \equiv \frac{B}{2}(y, -x) \quad \text{giving} \quad H = \hbar\omega_c\left(\frac{z\overline{z}}{4l^2} + \overline{z}\partial_{\overline{z}} - z\partial_z - 4l^2\partial_z\partial_{\overline{z}}\right) \tag{2.1.7}$$

In this gauge it is useful to have a finite circular geometry; we choose a two-dimensional disc as our QHE system, called an electron droplet. Wavefunctions in the LLL, $\phi_{0m} \equiv \langle z\ \overline{z}|0\ m\rangle$, should be annihilated by α and thus have the following form:

$$\alpha\phi_{0m} = -\frac{i}{\sqrt{2}}\left(\frac{z}{2l} + 2l\partial_{\overline{z}}\right)\phi_{0m} = 0 \qquad \implies \quad \phi_{0m} = f(z)e^{-\frac{z\overline{z}}{4l^2}} \qquad \text{if} \quad \partial_{\overline{z}}f(z) = 0 \tag{2.1.8}$$

The above condition is equivalent with the Cauchy-Riemann equations, implying that wavefuncitons in the LLL expressed in z and $\overline{z}$ consists of holomorphic functions with an additional factor $\exp(-|z|^2/4)$. This exponent is a recurring factor in QHE wavefunctions. An argument similar to (2.1.8) shows that ϕ_{00} is, apart from normalization, simply equal to this exponent. Furthermore we can find the form of different LLL wavefunctions:

$$\beta^\dagger\phi_{m0} = \frac{i}{\sqrt{2}}\left(-\frac{z}{2l} + 2l\partial_{\overline{z}}\right)f(z)e^{-\frac{z\overline{z}}{4l^2}} \sim z\phi_{m0} \tag{2.1.9}$$

Repeated use of $\beta^\dagger$ together with $\phi_{00} \sim \exp(-|z|^2/4l^2)$ then leads us to

$$\phi_{0m} = N_{0m} z^m e^{-\frac{z\overline{z}}{4l^2}} \tag{2.1.10}$$

where N_{0m} is a normalization factor. In the same manner one may show that the general wavefunctions are

$$\phi_{nm} = N_{nm} z^{m-n} L_n^{m-1}(z\overline{z}) e^{-\frac{|z|}{4l^2}} \tag{2.1.11}$$

where $L_a^b(z)$ are the generalized Laguerre polynomials, so that n gives the highest power of $\overline{z}$ and m the power of the single z-factor in the wavefunction.

Next we examine angular momentum:

$$\begin{aligned} L = \mathbf{r} \times \mathbf{p} &= \hbar(z\partial_z - \overline{z}\partial_{\overline{z}}) \\ \implies \quad L\phi_{nm} &= N_{nm}\hbar(z\partial_z - \overline{z}\partial_{\overline{z}}) z^m \overline{z}^n e^{-\frac{z\overline{z}}{4l^2}} = (m-n)\hbar\phi_{nm} \end{aligned} \tag{2.1.12}$$

Thus we know the significance of both quantum numbers; n gives the LL while $m - n$ is the angular momentum. Since β annihilates states ϕ_{n0}, this means that the possible angular momenta in a given LL are $j = -n, -n+1, \ldots$ The states are rotationally symmetric (up to an unimportant phase) as expected for angular momentum eigenstates but we can find the expectation value of the squared radius $r^2 = z\overline{z}$:

$$\begin{aligned} z = \sqrt{2}i(\alpha - \beta^\dagger) \qquad & \overline{z} = \sqrt{2}i(\beta - \alpha^\dagger) \\ \implies \quad \langle r^2 \rangle = |\tilde{N}_{nm}|^2 \langle 0\ 0 | \beta^m \alpha^n\ z\overline{z}\ & (\beta^\dagger)^m (\alpha^\dagger)^n | 0\ 0 \rangle = 2l^2(m+n+1) \end{aligned} \tag{2.1.13}$$

where $\tilde{N}_{nm}$ is normalization and we have used the orthogonality of states, annihilation properties and $\alpha^n \alpha^\dagger = n\alpha^n + \alpha^\dagger \alpha$.

This leads to a semiclassical picture of the electron droplet where, for each LL, the probability maxima of the states lie in concentric circles with increasing angular momentum out from the center. If we approximate the area with $\pi\langle r^2 \rangle$ and picture a maximum density droplet containing N_{LL} electrons in the each LL (that is, we fill the level with states having angular momentum from $-n$ to $N_{LL} - n - 1$) we find a total area per LL of $2\pi l^2 N_{LL}$, or an area of $2\pi l^2$ per state. We then introduce the filling factor ν which is the ratio of occupied states to available states per LL:

$$\nu = \frac{A}{2\pi l^2} = \frac{\Phi}{\Phi_0} \tag{2.1.14}$$

where $\Phi_0 \equiv h/e$ is the magnetic flux quantum and Φ is the magnetic flux through the area A. This means that the filling factor equals the number of flux quanta penetrating the system and that the density of the system decreases with increasing magnetic field and number of electrons kept constant.

We note that integer ν signifies ν completely filled LL's (in our geometry a disc inside the area $N2\pi l^2$ filled ν times). This is a non-degenerate state where ν LL's are populated with electrons having an angular momentum from n to $N - n - 1$. We will now examine the $\nu = 1$ state. The many-particle Hilbert space is spanned by a direct product of single particle Hilbert spaces, and the non-degenerate many-particle wavefunction is unique and independent of potential. It is the only antisymmetric linear superposition of N LLL wavefunctions having

angular momentum from 0 to $N-1$ and can be obtained using a Slater determinant (which is an antisymmetrized sum where every term has one element from each row):

$$\Psi_{\nu=1}(\{z_i\}) \equiv \langle\{z_i\}|\nu=1\rangle = \begin{vmatrix} \phi_{00}(z_1) & \phi_{00}(z_2) & \dots & \phi_{00}(z_N) \\ \phi_{01}(z_1) & \phi_{01}(z_2) & \dots & \phi_{01}(z_N) \\ \vdots & & & \vdots \\ \phi_{0N-1}(z_1) & \phi_{0N-1}(z_2) & \dots & \phi_{0N-1}(z_N) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ \vdots & & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \dots & z_N^{N-1} \end{vmatrix} e^{-\sum_i^N \frac{|z_i|^2}{4l^2}} = \prod_{i<j}^{N}(z_i - z_j)e^{-\sum_i^N \frac{|z_i|^2}{4l^2}} \tag{2.1.15}$$

where Ψ_1 is unnormalized. The proof of the last step is as follows: Multiply each row in the determinant with z_1 and subtract from the one below it. If we then expand this by the first column we get:

$$\begin{vmatrix} z_2 - z_1 & \dots & z_N - z_1 \\ z_2^2 - z_2 z_1 & \dots & z_N^2 - z_N z_1 \\ \vdots & & \vdots \\ z_2^{N-1} - z_2^{N-2} z_1 & \dots & z_N^{N-1} - z_N^{N-2} z_1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_2 & z_3 & \dots & z_N \\ \vdots & & & \vdots \\ z_2^{N-2} & z_3^{N-2} & \dots & z_N^{N-2} \end{vmatrix} \prod_{i=2}^{N}(z_i - z_1) \tag{2.1.16}$$

and we recognize the determinant above as the Slater determinant of particles 2 to N. The proof is obtained by repeated use of this process.

2.2 Gaps and incompressibility

The plateaus at integer and fractional ν also in the thermodynamic limit indicate that the system is gapped, in that a small but finite change in the magnetic field does not affect the resistivity. At integer ν there is indeed an energy gap, namely the cyclotron energy needed to reach the next LL. Disorder removes the energy gap but as we will see in the next section a mobility gap remains, because the low lying excitations are localized and do not contribute to electric transport. This concept of gaps is related to incompressibility:

The compressibility κ in two dimensions is defined (with total energy U, area A, pressure P, N particles and $N = nA$) as

$$\kappa \equiv -\frac{1}{A}\frac{\partial A}{\partial P}. \tag{2.2.1}$$

Using the thermodynamic relations

$$P = -\left(\frac{\partial U}{\partial A}\right)_{S,N,B} \quad \text{and} \quad \mu = \left(\frac{\partial U}{\partial N}\right)_{S,A,B} \tag{2.2.2}$$

We find

$$\kappa^{-1} = -A\frac{\partial P}{\partial A} = A\frac{\partial^2 U}{\partial A^2} = An^2\frac{\partial^2 U}{\partial N^2} = An^2\frac{\partial}{\partial N}\mu = n^2\frac{\partial \mu}{\partial n} \tag{2.2.3}$$

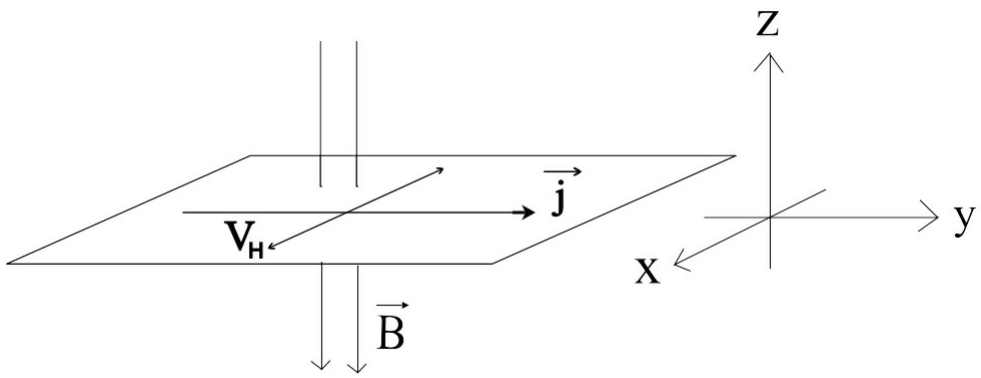

Figure 2.1: The quantum Hall effect in a rectangular geometry

A mobility gap means that the chemical potential increases discontinuously with particle number[2], resulting in $\frac{\partial \mu}{\partial n} \to \infty$. The above equation shows that this implies $\kappa = 0$ and thus incompressibility. As argued above integer ν's have a gap and therefore they are incompressible. Intuitively this can also be argued as follows: it is not possible to change the area per state infitesimally because this is constant in a given LL, and reaching the next LL requires a finite energy.

Thus the fractional ν where the QHE is observed must also be incompressible. This is harder to show because it requires taking interactions into consideration. We will examine the FQHE in detail later in the text.

2.3 The integer effect

Because the resistivity was experimentally shown to be different than what we argued in section 1.2 some of our assumptions must have been wrong. The upshot is that in a real sample we do not have translational invariance due to disorder and the relativistic derivation is erroneous. As we will see the disorder is in fact crucial to the occurence of the QHE.

The IQHE occurs when the disorder potential is sufficiently stronger than the Coloumb interactions, enabling a description where we neglect the latter. In order to show how a mobility gap occurs and study other aspects of the IQHE we introduce the Landau gauge:

$$\mathbf{A}_{Lan} \equiv -By\mathbf{e}_x \qquad \text{giving} \qquad H = \frac{1}{2m}\left(p_x^2 + (p_y - eBx)^2\right) \tag{2.3.1}$$

A natural geometry for this gauge (which also resembles the classical Hall effect setup) is a rectangular system where the current is in the y-direction and the Hall voltage is between the confining edges in the x-direction (see figure 2.1). We assume that the confining potential does not break the translational symmetry in the y-direction so that the Hamiltonian commutes with canonical y-momentum . Then we can use separation of variables to obtain wavefunctions

[2]This is because a gap implies a difference between the energy increase when a particle is added and the decrease when a particle is removed, so that a particle-hole pair, which is the lowest excitation, costs a finite energy to produce.

that are eigenstates of p_y, that is plane waves:

$$H\psi(x,y) = \frac{1}{2m}\left(p_x^2 + (p_y - eBx)^2\right) f(x)e^{ik_y y} = \frac{1}{2m}\left(p_x^2 + (k_y - eBx)^2\right) f(x)e^{ik_y y}$$
$$\implies \quad Hf(x) = \frac{1}{2m}\left(p_x^2 + m\omega_c^2(x + k_y l^2)^2\right) f(x) \equiv E_k f(x) \tag{2.3.2}$$

Again we recognize the above as equivalent to a (displaced) Harmonic oscillator system in the x-direction, yielding the LL's and with displaced Hermite polynomials as solutions: $\psi(x,y) \propto H_n(x+k_y l^2)\exp(ik_y y)$. While the states are extended in the y-direction these polynomials have the effect that the x-coordinate is quasi-localized to $x = -k_y l^2$, so that in this system the y-momentum eigenvalue fixes the x-position. With periodical boundary conditions and a length of L_y in the y-direction we find that the possible momentum eigenvalues are $k_y = 2\pi j/L_y$ where j is an integer. Using $x \in [0, L_x]$ we may then count the number of states per LL[3] with j:

$$\Delta j = \frac{k_y^{max} - k_y^{min}}{2\pi/L_y} = \frac{L_x L_y}{2\pi l^2} \qquad \implies \qquad \nu = \frac{\Delta j}{A} = \frac{1}{2\pi l^2} \tag{2.3.3}$$

where the area $A = L_x L_y$ and we recognize the expression for the filling factor.

Similar to our treatment in the symmetric gauge we look at operators corresponding to the classical guiding centers:

$$C_x \equiv x + \frac{\Pi_y}{m\omega_c} \qquad C_y \equiv y - \frac{\Pi_x}{m\omega_c} \tag{2.3.4}$$

Again one may show that small wavepackets execute cyclotron motion around $\mathbf{C}$ when there is no potential. We then assume that the motion of wavepacks will still be centered around these operators when including a potential arising from disorder and the confining edges (which is legitimate as long as the potential is relatively smooth) and thereby approximate the particle coordinate operators with $\mathbf{r} \to \mathbf{C}$. The Heisenberg equations with the altered Hamiltonian $H_V = H + V$ then give

$$i\hbar\dot{C}_x = [C_x, H_V] = [C_x, V] = \frac{dV}{dC_y}[C_x, C_y] = -il^2\frac{\partial V}{\partial C_y}$$
$$i\hbar\dot{C}_y = [C_y, H_V] = [C_y, V] = \frac{dV}{dC_x}[C_y, C_x] = il^2\frac{\partial V}{\partial C_x} \tag{2.3.5}$$

where we have used that $\mathbf{C}$ commutes with the original Hamiltonian H, the approximation $V(x,y) = V(C_x, C_y)$ and the following identity ($\mathcal{O}_j$ are operators):

$$[\mathcal{O}_1, f(\mathcal{O}_2, \ldots, \mathcal{O}_n)] = \sum_j \frac{df}{d\mathcal{O}_j}[\mathcal{O}_1, \mathcal{O}_j] \tag{2.3.6}$$

Assuming $[\mathcal{O}_1, [\mathcal{O}_1, \mathcal{O}_j]] = 0$. From (2.3.5) we see that

$$\dot{\mathbf{C}} \cdot \nabla_{\mathbf{C}} V = \frac{\hbar}{l^2}\left(\dot{C}_x\dot{C}_y - \dot{C}_y\dot{C}_x\right) = 0 \qquad \implies \qquad \dot{\mathbf{C}} \perp \nabla_{\mathbf{C}} V \tag{2.3.7}$$

[3]Each LL has different polynomials H_n in the x-direction and plane waves with $k_y \in [-x/l^2, 0]$ in the y-direction

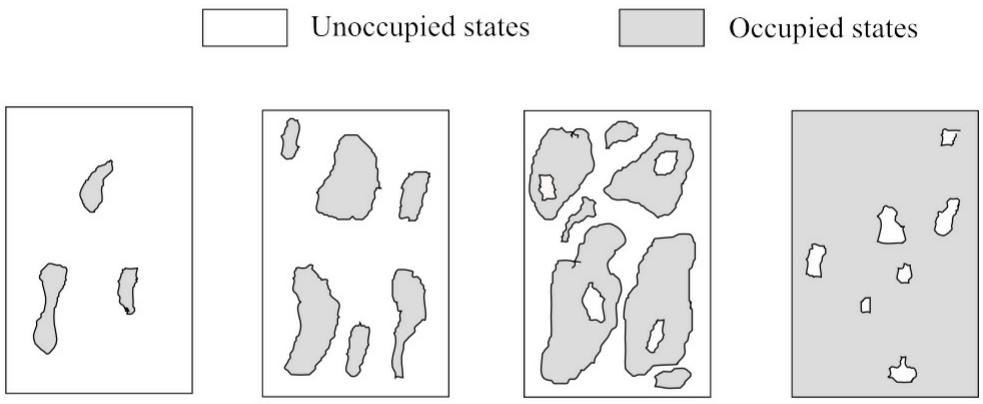

Figure 2.2: Occupation of states in the IQHE as the magnetic field changes

so that the guiding centres and thus the wavepacks move along equipotential lines. A heuristic argument for the same conclusion goes like this: with the magnetic field locking the kinetic energy to a certain value (as long as there is no LL mixing) and the total energy conserved the particles must have a constant potential energy and therefore move along lines with $\nabla V = 0$.

In the bulk of the sample this means that, as long as the equipotential lines do not stretch from one edge to the other (in the y-direction) the electrons are localized and do not contribute to electronic transport; leading to the advertised mobility gap. If we for example start at an integer ν and decrease the magnetic field (decreasing the number of available states per LL and forcing electrons to move to higher LL's) the added electrons in the highest LL will be confined to the potential lines (even repulsive ones; owing to the magnetic field freezing out kinetic energy). The edge potentials on the other hand do stretch from one perpendicular edge to the other. Thus each LL has a chiral, one dimensional current at the edges, going in opposite directions at the opposite edges because the potential gradients have different sign.

The particles in the bulk will occupy the lowest potentials in the landscape first, going higher and higher as the magnetic field rises until the potiential lines cross the sample. This is the transition between plateaus, and at this point the current from another LL is added to the flow along the edges so that we reach a new plateau with a higher conductivity. After this the story is repeated with holes until we reach the next integer ν. This process is illustrated in figure 2.2.

On the plateaus there is no backscattering leading to dissipation because this would entail tunneling from one edge to the other, a macroscopic distance. However as we are nearing the transition the formerly isolated equipotential lines are getting close to one another and the edges so that backscattering is possible, leading to dissipation.

The conclusion is that the disorder causes the LL's to broaden but that only the states with energy around the original levels carry current, see figure 2.3. With this we have explained the plateaus themselves but we can also derive the resistivity. In a somewhat semiclassical fashion we can calculate the current density resulting from each filled LL using the dispersion relation for direction i with wavepackets in mind:

$$\mathbf{e}_i \cdot \mathbf{w} = \frac{\partial E}{\partial k_i} \tag{2.3.8}$$

Where **w** is group velocity. To find the current from LL number n in the y-direction we sum $-ew_y/L_y$ over all states, which can be expressed as a phase space integral:

$$I_y^n = -\frac{e}{L_y}\int_0^{L_y} dy \int_{-\infty}^{\infty} \frac{dk_y}{h}\frac{\partial E_n}{\partial k_y}P_k = -\frac{e}{h}\int_{E(k_y=-\infty)}^{E(k_y=\infty)} dE_n P_E = \frac{e}{h}(\mu_R - \mu_L) \tag{2.3.9}$$

Where P is the probability of the state indicated by the index being occupied and we have taken advantage of the temperature limit $T \longrightarrow 0$. At zero temperature P_E is 1 for all energies below the chemical potential μ and 0 for all above, Thus P_E becomes a step function which picks out the energies at the integral limits to be equal to the chemical potentials of maximum and minimum k_y. Owing to the relation between k_y and x these are the chemical potentials at the left (μ_L) and right (μ_R) edges respectively. The difference between these is the energy required to bring one particle from R to L, which in this case is $\mu_L - \mu_R = eV_H$ where V_H is the Hall voltage, or the voltage in the x-direction. This is repeated for each of the ν LL's, giving a total current

$$I_y = \nu(-\frac{e}{h})(-eV_H) = \nu\frac{e^2}{h}V_H \quad \Longrightarrow \quad R = \rho = V_H/I_y = \frac{1}{\nu}\frac{h}{e^2} \tag{2.3.10}$$

Together with the observation that there is no current in the x-direction this gives the correct expression for ρ.

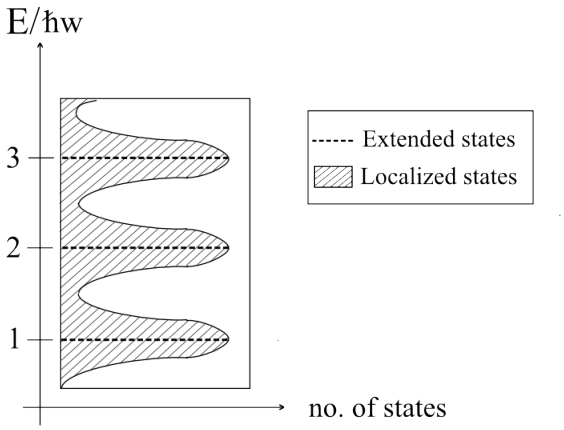

Figure 2.3: The energy spectrum of a QHE system with disorder

2.4 The gauge argument

Although the derivation in the previous section should explain the occurance of the IQHE another argument due to Laughlin [12] and Halperin [13] will be useful with later chapters in mind. Our starting point is the one shown in figure 2.3, where the impurities broaden the LL's but the states are localized except for those close to the original LL energies. Then we consider

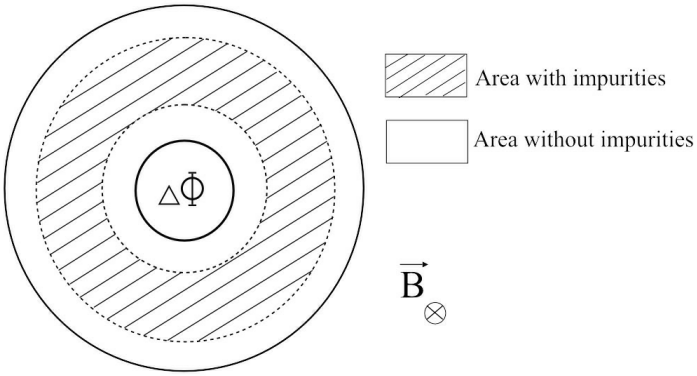

Figure 2.4: The Corbino disc thought experiment

a special disc-geometry with a hole in the centre called a Corbino disc (the universality of the QHE permits us to do this). We imagine the outmost section (closest to the outer physical edge) being ideal, without impurities, the middle section to be a physical QHE system and the inner section again being ideal (see figure 2.4). The usual perpendicular, strong magnetic field is present.

In the ideal sections we assume that an integer number of LL's are completely filled so that there are actual energy gaps and not just mobility gaps. Then the Fermi energi lies between LL's and thus in a region devoid of states. Since the sections of the disc are in thermodynamic contact the energy is the same in the one with impurities, thus lying in a region with localized states only.

The ideal sections are translationally invariant due to the absence of impurities so that the result in (1.2.4) is correct. Since the filling factor is $\nu = \rho/2\pi l^2$ we see from (1.2.4) that the off-diagonal components of the resistivity are $\pm h/\nu e^2$ - the relativistic derivation does hold for the general (non-invariant) case when ν is an integer.

Then we insert an additional perpendicular adiabatically changing magnetic flux Φ_C through the central hole (more on adiabatic change in section 4.2.2). This flux can be represented through the following vector potential[4]

$$\mathbf{A} = \frac{\Phi_C}{2\pi r}\mathbf{e}_\phi \qquad \text{since} \qquad \Phi_A = \int_\Omega (\nabla \times \mathbf{A}) \cdot d\mathbf{\Omega} = \oint_{\partial\Omega} \mathbf{A} \cdot d\mathbf{l} = \Phi_C \tag{2.4.1}$$

in polar coordinates (r, ϕ) where $d\mathbf{\Omega}$ is a directional infinitesimal area element of Ω and $d\mathbf{l}$ is an element of $\partial\Omega$, the boundary of Ω.

We see that this potential can be gauged away with $\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\Lambda$ if $\Lambda = \frac{\Phi_c \phi}{2\pi}$. In keeping with local gauge invariance this will on the other hand give a factor on the wavefunction:

$$\Psi \to \Psi' = e^{\frac{ie}{\hbar}\frac{\Phi_C \phi}{2\pi}}\Psi \tag{2.4.2}$$

[4] Φ_C gives a singular magnetic field where $\mathbf{B}$=0 except for at $r = 0$, but this should not matter for our use.

where the change will only affect single-particle states that encircle the hole (see the discussion on the Aharonov-Bohm effect in section 4.2, of which this is an example). But if we remove the solenoid after a single magnetic flux quantum $\Phi_0 = h/e$ has been inserted the resulting factor is $\exp(2\pi i)$ - an unobservable shift. Then, since the change was adiabatic, we are back to the original configuration with no vector potental and the initial eigenstate wavefunction.

But from the expression for the resistivity $\rho_{\mu\nu}$ we see that the azimuthal electric field $\mathbf{E}$ induced by the changing magnetic flux must result in a radial current transporting a total charge of

$$Q = \int dt I = \int dt \int \mathbf{j} \cdot d\mathbf{l} = \int dt \oint_{\partial\Omega} \sigma \mathbf{E} \cdot d\mathbf{l} =$$
$$\sigma_{01} \int dt \int_\Omega (\nabla \times \mathbf{E}) d\Omega = -\sigma_{01} \int dt \int_\Omega \frac{\partial \mathbf{B}}{\partial t} d\Omega = -\sigma_{01}\Phi_0 = \nu e \tag{2.4.3}$$

where I is the total current. We have used that the conductivity tensor σ (from which the component 01 is the only contribution) is the inverse of the resistivity and Faradays induction law. The only result of the flux insertion is then that an integer number of electrons are transported into the physical section containing impurities.

These electrons must come from the outer physical edge, whose gapless states are filled to the Fermi energi because of the low temperature. This property should persist after the flux change, meaning that the energies of the electrons entering the physical section are at the Fermi level.

But the states in this section which have the Fermi energy can as we saw not be affected by the flux change since they are localized. Thereby no change in their occupation is allowed [10] and the electrons must pass through the physical region to the inner one. This means that the transport properties of the gapped ideal sections and the one containing impurities are the same- In other words, providing the Fermi energi lies in a mobility gap the resistivity tensor is given by

$$[\rho_{\mu\nu}] = \frac{1}{\nu}\frac{h}{e^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.4.4}$$

and the plateaus in resistance vs. magnetic field appear.

We will now redirect our attention from the IQHE to the FQHE, but first we take a detour to look at the concept of anyons.

Chapter 3

Anyons

This chapter serves as a general introduction to anyons, a key concept for our purposes.

3.1 The configuration space

There are two ways of looking at the statistics of identical particles, namely exchange statistics and exclusion statistics. The former concerns the phase factor picked up by the wavefunction when two particles are interchanged; the subject of this text. The latter was generalized to anyons by Haldane [14] and deals with occupation of states. If a fermion occupies a given state there is exactly one less state available while for bosons there is no change. Anyons interpolate between this and block a number of states S ranging between one and zero. If for example $S = 1/3$ a single state can accomodate 3 anyons. The Pauli principle is a manifestation of exclusion statistics in three dimensions: Since the fermion wavefunction has a sign change after interchange one may show that the probability of two fermions occupying the same state is zero. Bose-Einstein condensation on the other hand depends on the bosons having no similar limit.

Thus there is a close connection between the two viewpoints on statistics in three dimensions. In the two-dimensional case we find a generalization of both, with exchange statistics angles interpolating between 0 and π and exclusion statistics ranging from bosonic to fermionic. Although closely related the two descriptions are not a priori equivalent. We will not examine exclusion statistics any further.

The usual textbook argument concerning exchange statistics goes something like this: If two particles are fundamentally indistinguishable the physical system should be unchanged if they swap places, implying that the maximum change in the wavefunction is a phase factor (assuming normalization before and after). Thus, with an interchange operator O we have $O\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{i\theta}\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_2, \mathbf{r}_1)$. The angle θ is called the statistics angle. Moreover, two interchanges bring the system identically back to its original configuration, so that we have

$$O^2\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{2i\theta}\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_1, \mathbf{r}_2) \qquad \Longrightarrow \qquad \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots \tag{3.1.1}$$

Since a phase difference of 2π is trivial this means that we have two possibilities: a phase change of zero (bosons) or π (fermions).

But in 1977 Leinaas and Myrheim analyzed this argument more carefully [15], employing a physical exchange process rather than an abstract operator O acting on particle indices with unclear physical meaning. There was then no need to impose additional symmetry constraints

as in the former argument. They found that in dimensions $d > 2$ the result is the same, but not in one and two dimensions - implying a theoretical possibility of the existence of particles called anyons.

Firstly we must bear in mind that the space the (multi-particle) wavefunction is defined on is the configuration space of the particles, which is not necessarily Cartesian but can have a more complicated topology. If this is the case, and especially if there are singularities, we must be careful. Indeed identical particles lead to a nontrivial topology[1]: If the single-particle configuration space is the d-dimensional Euclidian space $\mathcal{E}_d$ then the configuration space $\mathcal{C}_N$ of N identical particles is not the Cartesian product $\mathcal{E}_d^N \equiv \mathcal{E}_d \otimes \mathcal{E}_d \otimes \cdots$, but rather

$$\mathcal{C}_N = \mathcal{E}_d^N / S_N \tag{3.1.2}$$

which means that one should divide out the action of the permutation group S_N, or in other words that points in $\mathcal{E}_d^N$ where particles are simply interchanged should be identified. Since the group S_N is discreet and finite $\mathcal{E}_d^N / S_N$ is locally isomorphic to $\mathcal{E}_d^N$, but the points of coinciding particles are singular [15]. In the classical case it is sufficient to ignore this distinction and divide the configuration space by $N!$ to remove multi-counted configurations (cf. Gibb's paradox), but the quantum description is more complicated.

We see that $\mathcal{C}_N$ can be simplified by considering center of mass and relative degrees of freedom. The center of mass space is clearly a simple Euclidian space: $\mathcal{C}_N = \mathcal{E}_d \otimes \mathcal{R}_{d,N}$ where $\mathcal{R}_{d,N}$ is the relative space of N particles in d dimensions. The topology of $\mathcal{R}$ differs with unequal dimensionality. As mentioned $d > 2$ reduces to the familiar case of bosons and fermions, while the $d = 2$ case gives rise to the concept of anyons. A qualitative explanation of this goes as follows:

We imagine two particles swapping places by encircling each other in a counterclockwise sense. Can this operation be physically distinguished from the corresponding one in a clockwise sense? In three dimensions the answer is no because we can simply rotate the system by 180°. Therefore two interchanges are equal to none at all (rotate one of them), and the argument in the first paragraph of this chapter applies. But when we are restricted to two dimensions we cannot perform this rotation because we would need the third direction. Thus trajectories involving any number of interchanges can in principle be physically distinguished from each other. In the next section we take a quantitave look at the two dimensional two-particle case.

3.2 2 dimensions

The configuration space for two particles in two dimensions is $\mathcal{C}_2 = \mathcal{E}_2 \otimes \mathcal{R}_{2,2}$. The interesting part is $\mathcal{R}$, which in this case is a two-dimensional Euclidian space with points of interchanged particles identified. The relative coordinate is $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ where $\mathbf{r}_i$ is the single-particle coordinate of the i'th particle. This means that an interchange implies $\mathbf{r} \to -\mathbf{r}$ and thus these points are equal. The resulting topology of $\mathcal{R}$ can be visualized as follows: We start with the Euclidian plane and make a cut along a line l_1 from the origin. Then we fold the plane around so that one end of the cut coincides with a second line l_2 which is l_1 reflected through the origin (figure 3.1). The result is a cone with an angle of 30° between a line through the

[1] This route to the anyon concept, which is the same as the one emplyed by Leinaas and Myrheim, applies the consequences of identical particles to the configuration space. Another one introduced by Wilczek [16] entails altering the potential.

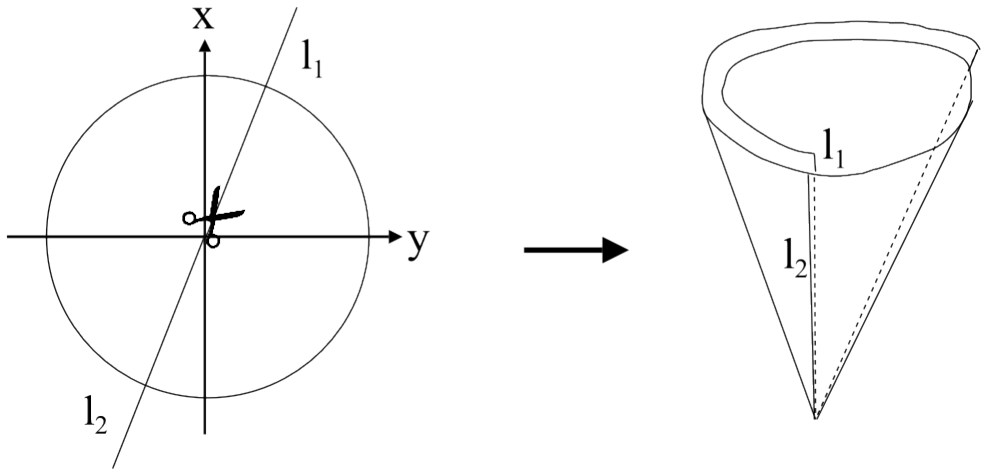

Figure 3.1: Visualization of the two-dimensional two particle configuration space

middle of the cone and the outer boundary.[2]

This space is locally flat except for a singularity at the tip of the cone, which is the origin and thus where particles coincide. Now we may consider closed particle trajectories which encircle the singularity a number of times ranging from $-\infty$ to ∞ - we dub this integer the winding number w (with positive w denoting counterclockwise orbits and negative w denoting clockwise). Since these paths are closed on the cone they correspond to two types of trajectories in the original plane: closed ones or those reaching from $\mathbf{r}$ to $-\mathbf{r}$, with the particles respectively either ending up at the same place or interchanging (since $\mathbf{r}$ is the relative coordinate). Thus w counts the number of times the two particles are interchanged. The important observation is that paths with different w cannot be continuously deformed into each other without crossing the singularity. Different particles may a priori behave differently when encountering a singularity in $\mathcal{C}_N$, and the upshot is that w gives a way of physically distinguishing between various particle trajectories.

A space in which every closed path can be continuously deformed into one another is called simply connected, while if this is not the case it is called multiply connected. This means that $\mathcal{R}_{2,2}$ is multiply connected. Paths that cannot be changed into each other in this way are said to be in different homotopy classes.[3]

To explore the consequences of this we use Feynman's formalism of path integrals. In the latter two-point correlation function are expressed as a sum/integral of a phase factor over all possible trajectories in the configuration space. The amplitude for a particle to travel from z

[2]We imagine drawing a circle of radius a in the original Euclidian plane; after the folding it twice encircles a disc normal to the line through the middle of the cone. Then a triangle consisting of the latter line, the radius of the disc and the outer boundary up to the disc has a shortest side of length $a/2$ and a hypotenuse of length a so that the desired angle is $30°$.

[3]Homotopy is related but not equivalent to homology, which we considered in section 1.5. For example the letters O and P are in the same homotopy class but not homologic.

to z' in the time interval from t to t' is given as

$$A(z',t';z,t) = \int_z^{z'} \mathcal{D}z\mathcal{D}\overline{z}e^{i\int_t^{t'} L(t_1)dt_1} \tag{3.2.1}$$

where L is the Lagrangian and $\int \mathcal{D}z$ signifies the integral of all possible paths betwen the integral limits (this is often difficult to give a mathematical formulation but that is unneccessary in our context). Ignoring the trivial center of mass motion we take z to be the relative coordinate of the two particles. As emphasized above we may divide the paths into classes corresponding to different winding numbers w so that we have

$$A(z',t';z,t) = \sum_{w=-\infty}^{\infty} A_w(z',t';z,t) \tag{3.2.2}$$

Since $\mathcal{R}$ is multiply connected we may then generalize A with complex weights according to winding number, although within a specified homotopy class we cannot add relative weights between different paths. This is because consistency demands that the contribution from a given path must continuously transform into any other when the paths are also continuously deformed into each other. However if this can't be done (as is the case with paths from different homotopy classes) there is no a priori reason why we can't have such weights. It can be shown that these can only be phases [17]. Denoting position with polar coordinates $z = re^{i\chi}$ we thus may write:

$$\widetilde{A}(r',\chi',t';r,\chi,t) = \sum_{w=-\infty}^{\infty} C_w A_w(r',\chi',t';r,\chi,t) \tag{3.2.3}$$

where C_w is a phase factor, $A_w(r',\chi',t';r.\chi,t) = \int_{z_w} \mathcal{D}z\mathcal{D}\overline{z}e^{iS}$ and z_w signifies paths with w windings. From here on we assume that the starting and ending configurations are equal except for winding number so that $r' = r$. Now, if we continuously change $\chi' \to \chi' + \pi$ we have the same physical situation and a phase factor θ is the only possible effect. But we also have $A_w(r,\chi'+\pi,t';r,\chi,t) = A_{w+1}(r,\chi',t';r,\chi,t)$, yielding

$$\begin{aligned}\widetilde{A}(\chi') = \sum_w C_w A_w(r,\chi',t';r,\chi,t) = e^{i\theta}\widetilde{A}(\chi' \to \chi'+\pi) \\ = e^{i\theta}\sum_w C_w A_{w+1}(r,\chi',t';r,\chi,t) = e^{i\theta}\sum_w C_{w-1}A_w(r,\chi',t';r,\chi,t)\end{aligned} \tag{3.2.4}$$

If we equate coefficients of A_w, giving a recurrence relation for C_w, and choose $C_0 = 1$ this gives us

$$\widetilde{A}(z',t';z't) = \sum_{w=-\infty}^{\infty} e^{iw\theta}A_w(z',t';z,t) \tag{3.2.5}$$

Which shows that w anticlockwise/clockwise interchanges of particles gives a phase factor $\exp(iw\theta)$ where w is positive/negative on the correlation function. The statistics angle θ depends on the type of anyon; in the FQHE we will get rational multiples of π. Heuristically one can say that the strength of the singularity at the relative position zero decides the angle; bosons can be on top of each other and thus have $\theta = 0$ while fermions are strongly repelled cf. the Pauli principle and have $\theta = \pi$.

To find the consequences for wavefunctions in the Schrödinger picture we recall the expression for a propagator (assuming that the Hamiltonian is time-independent):

$$A(z',t';z,t)=\int \mathcal{D}z\mathcal{D}\bar{z}e^{iS}=\langle\psi(z,t)|\psi(z',t')\rangle=\langle\psi(z,0)|e^{-\frac{i}{\hbar}H(t'-t)}|\psi(z',0)\rangle \tag{3.2.6}$$

To separate into sub-propagators with differing winding number as previously we use multivalued wavefunctions. Then we may associate different functions $|\psi_w\rangle$ with differing w, although they in fact represent the same particle configuration. Since each propagator A_w is a genuine solution to the Schrödinger equation we thus may write

$$\widetilde{A}(z',t';z,t)=\sum_{w=-\infty}^{\infty}e^{iw\theta}A_w(z',t';z,t)=\sum_{w=-\infty}^{\infty}\langle\psi(z,t)|\psi_w(z',t')\rangle \tag{3.2.7}$$

Then $\widetilde{A}$ gives the probability amplitude for the particles to end up at the same position regardless of their trajectories between t and t', while the various $\psi_w(z,t)=\langle z|\psi_w(t)\rangle$ describe the system after different interchanges[4]. It is natural to associate a path integral having a given number of windings with the corresponding wavefunction. Since the various $|\psi_w\rangle$ represent the same physical situation apart from motion around the relative space singularity they should not differ with any relative phases except for the ones arising from their multivaluedness. From inspection of the above equation we therefore conclude:

Each counter-clockwise interchange of anyons gives a factor $e^{i\theta}$ on the wavefunction. (3.2.8)

The last equality in (3.2.6) shows that we may also leave the wavefunctions singlevalued and instead incorporate the effects of the singularity as extra terms in the Hamiltonian (as mentioned earlier), a method called flux tube attachment. Thus two equivalent Schrödinger representation descriptions of anyons exist. Although we will stick to multivalued functions the arguments leading up to these are often strongly reminiscient of flux attachment.

The crucial difference between two and three dimensions is the connectedness of $\mathcal{R}$. In three dimensions paths can be deformed into each other to a larger extent, resulting in only two distinct classes, in contrast with the unbounded number of possibilities when $d=2$. In other words, $\mathcal{R}_{3,2}$ is doubly connected and gives rise to bosons and fermions while $\mathcal{R}_{2,2}$ is infinitely connected and yields anyons. Considering two particles is sufficient to our needs and we will not look at the N particle case.

[4]This may sound like it implies the existence of a unique starting configuration after which w counts the windings of the paths. But since we only need to define relative number of interchanges between an arbitrary initial state and an ending state this is not the case.

Chapter 4

Laughlin's wavefunctions and the hierarchy

In light of the discussion in section 2.2 there is no reason to expect plateaus at fractional fillings, because these states are degenerate and have no apparent energy gap. We have not yet seen a way to create incompressibility at fractional ν, but the fact that the FQHE only occurs at very pure samples is a hint that the missing mechanism lies in the interactions between the electrons. What makes the FQHE special is that the usual approximative approach is impossible - we cannot first solve the problem for free particles and then use a perturbative expansion in a given parameter to include interactions. This is because in our (to a certain degree ideal) system the only term in the Hamiltonian *is* the interaction term, the remaining removed due to polarization (Zeeman term), restriction to a single LL (kinetic energy term, reduced to being equal for all particles and thus irrelevant) and neglecting impurities (disorder potential, which is unimportant in this context except to provide localization).

To solve the N-particle problem with Coloumb interactions is out of the question (it can be done numerically for small systems but this does not give much insight into the physics). An approach that has been successful in FQHE physics is to make educated guesses on trial wavefunctions, comparing these numerically to the exact solution and studying their properites. The remainder of the thesis is centered on this method.

4.1 Laughlin's wavefunctions

Laughlin created a set of trial wavefunctions [18] in the symmetric gauge for FQHE states lying in the LLL with $\nu = 1/q$, $q = 1, 3, 5, \ldots$, called the Laughlin sequence. They are modifications of the exact $\nu = 1$ wavefunction Ψ_1 encountered in (2.1.15), namely

$$\Psi^L_{1/q} \equiv \prod_{i<j}^{N} (z_i - z_j)^q e^{-\sum_i \frac{|z_i|^2}{4l^2}} \tag{4.1.1}$$

where the above is unnormalized; Ψ^L has not been normalized in closed form in the literature. We see that by restricting q to odd integers the antisymmetry of the electrons is kept intact. There are several motivations for this guess: Ψ^L is holomorphic aside from the exponential, apart from a constant factor translationally invariant (breaking of this symmetry would lead to gapless Goldstone boson excitations and thus compressibility), it has a mobility gap (with

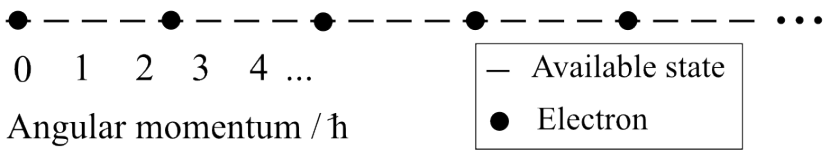

Figure 4.1: Graphic representation of $\nu = 1/3$

localized qp's instead of electrons as seen below), it has the correct filling factor, it is the zero energy eigenstate of an approximate potential (see below), it has an extremely good numerical overlap with the exact functions and it is consistent with the plasma analogy which we will introduce later. We expand upon some of these points:

Because electrons are fermions they have a natural tendency to avoid each other, as can be seen from Ψ_1: The wavefunctions tend toward zero when $z_i \to z_j$. This can be expressed by evoking the concept of a correlation hole around each electron. The correlation function can be described as giving the probability of finding other particles at different points when we assume that a single particle is at a given point, and this will be small in a region around a fermion - it is surrounded by a hole. We see that $\Psi^L_{1/q}$ has a larger correlation hole than Ψ_1 due to the exponent q, causing the function to shrink faster when particles approach. Therefore the system described by Laughlin's wavefunctions is more sparse than the $\nu = 1$ case, which is consistent with the lower filling factor. The sparsity is also in keeping with the repulsive interactions of the electrons.

Laughlin's wavefunction is an eigenfunction of relative angular momentum between the electrons, each eigenvalue being $j = q$, and a superposition of different distributions of single-particle angular momenta (the total single-particle angular momentum on the other hand is well defined). Every term in the superposition has particles with angular momentum ranging from 0 to jN, which can be seen from the expansion

$$\prod_{i<j}^{N}(z_i - z_j)^q = z_1^{qN} z_2^{q(N-1)} \cdots z_{N-1} + \ldots \tag{4.1.2}$$

In analogy with the arguments leading up to (2.1.14) this indicates that the total area of the LLL is $2\pi l^2 Nq$ so that we get the expected $\nu = \frac{1}{q}$ (since we find $1/q$ of the area of a single LL and a filled LLL gives $\nu = 1$). A graphic picture of the filling factor $\nu = 1/3$ is displayed in figure 4.1, where a line represents an angular momentum state and a dot indicates an electron[1].

The aforementioned approximate potential is that of the hard-core model. If we assume that the interaction potential is central we may decompose it into two-particle Haldane pseudopotentials $v_j = \langle j|V|j\rangle$ where j is relative angular momentum (the interaction term acts only on the relative-motion degrees of freedom) in the following way:

$$V = \sum_{k<l}^{N} V(|z_k - z_l|) = \sum_{k<l}^{N}\sum_{j=0}^{\infty} v_j P_j^{kl} \tag{4.1.3}$$

[1]The actual system is a superposition of different placings of these particles but the one in the figure is the most prominent because of the correlation holes.

where P_j^{kl} projects the pair k, l onto relative angular momentum j. Raising j entails moving particles further from each other and thus gives a lower energy from the repulsive potential. Motivated by this the hard-core model is obtained by approximating $v_j = 0$ for $j > 0$ so that we get $V_{hc} = \sum_{k<l}^{N} v_0 P_0^{kl}$. We see immediately that Laughlin's wavefunctions are zero-energy eigenstates of this potential.

4.2 Laughlin quasiparticles

When we shift the magnetic field slightly away from the value giving exactly $\nu = 1/q$ we move along the plateau in the resistivity-magnetic field curve. To see what happens in this case we can imagine inserting an infinitely thin solenoid into the electron droplet and slowly increase the resulting magnetic flux, with the direction of the current deciding whether this will increase or decrease the total flux. The flux change is assumed to be adiabatic, which can be described as slow enough that the system stays in the same (changing) energy state; see section 4.2.2. At this point we need to look at a strictly quantum mechanical phenomenon called the Aharonov-Bohm effect.

To compute this effect we follow [19] and look at an altered double-slit experiment in the path integral formulation. Initially we consider the ordinary double slit setup (see figure 4.2) and concentrate on two arbitrary paths $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ between the starting point and a given point at the far side, where the paths go through different slits. Their contributions to the amplitude for a particle to be registered at the endpoint is

$$e^{\frac{i}{\hbar}S[\mathbf{p}_1]} + e^{\frac{i}{\hbar}S[\mathbf{p}_2]} = e^{\frac{i}{\hbar}S[\mathbf{p}_1]}\left(1 + e^{\frac{i}{\hbar}(S[\mathbf{p}_1]-S[\mathbf{p}_2])}\right) \tag{4.2.1}$$

where S is the action, so that the paths may interfere constructively or destructively depending on the relative phase $(S[\mathbf{p}_2] - S[\mathbf{p}_1])/\hbar$. We now imagine having a magnetic flux between the two paths, restricted so that the magnetic field does not actually reach the particles (meaning that in a classical situation it would have no effect on their trajectories). However the vector potential in $\mathbf{B} = \nabla \times \mathbf{A}$ (usually thought of as a mathematical construct only) may have contact with the paths. Through the Lagrangian the action is then altered:

$$\Delta S = \int dt \Delta L = -Q \int dt \frac{d\mathbf{p}}{dt} \cdot \mathbf{A} = -Q \int d\mathbf{p} \cdot \mathbf{A} \tag{4.2.2}$$

where we write $S' = S + \Delta S$ and Q is the particle charge.

If we now look at the total amplitude we get

$$\begin{aligned} A' = \int \mathcal{D}\mathbf{p} e^{\frac{i}{\hbar}S'} = \int_{slit\ 1} \mathcal{D}\mathbf{p} e^{\frac{i}{\hbar}S'} + \int_{slit\ 2} \mathcal{D}\mathbf{p} e^{\frac{i}{\hbar}S'} = \\ e^{-\frac{iQ}{\hbar}\int d\mathbf{p}_1 \cdot \mathbf{A}} \left(A_1 + e^{-\frac{iQ}{\hbar}\left(\int d\mathbf{p}_1 \cdot \mathbf{A} - \int d\mathbf{p}_2 \cdot \mathbf{A}\right)} A_2\right) \end{aligned} \tag{4.2.3}$$

where A_1 and A_2 are the amplitudes for the particle going through slit 1 or 2 respectively without the magnetic field and we have used that all line integrals for paths passing through a given slit and arriving at the same point are equal. But this means that we get a relative phase difference between contributions (which thus have an observable effect) arising from the magnetic field only, equal to

$$\phi_{AB} = \frac{Q}{\hbar}\left(\int d\mathbf{p}_2 \cdot \mathbf{A} - \int d\mathbf{p}_1 \cdot \mathbf{A}\right) = \frac{Q}{\hbar} \oint d\mathbf{p} \cdot \mathbf{A} = \frac{Q\Phi}{\hbar} \tag{4.2.4}$$

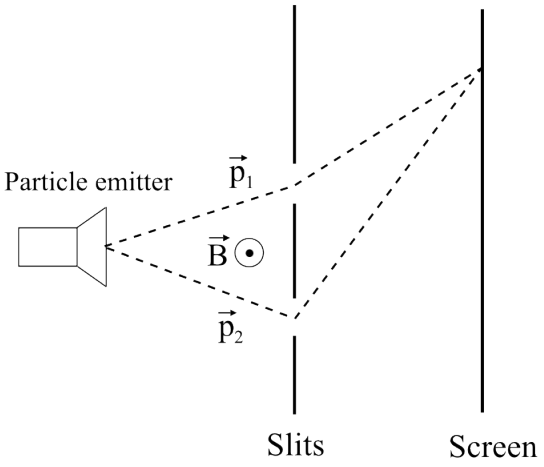

Figure 4.2: Double slit setup with magnetic field

where Φ is the magnetic flux between the paths. The phase is negative or positive depending on the direction of the magnetic field.

We now return to our solenoid. Though it does not touch any electrons it will alter their Hamiltonians through the Aharonov-Bohm effect. Since the change is adiabatic the system will remain an eigenstate of the changing Hamiltonian. The potential used in section 2.4 is suitable, and in analogy with that calculation we see that when the flux through the solenoid has reached $\Phi_0 = h/e$ we may use a gauge transformation to remove it from H. This means that the system is in an eigenstate of the original Hamiltonian, but a different one than initially. Since the ground state is non-degenerate the system is now in an excited state. We have increased or decreased the flux through the system with one unit, creating an unoccupied or removing an occupied state (in the latter case the state with lowest angular momentum will be transferred to the next LL [20]. This indicates the creation of an entity we call a quasiparticle; quasiholes (qh) in the case of missing electrons and quasielectrons (qe) in the case of additional ones.

The arguments in section 2.4 show that the expression for the QHE resistivity should hold generally within the bulk, independent of edges. Also, in a similar way as the flux in that section, the solenoid will induce a radial current in the electron droplet. Following the same argument and integrating along a circle containing the qp, the charge resulting from the solenoid is

$$Q_{qp} = \pm \nu e \tag{4.2.5}$$

with the sign depending on the current through the solenoid. But now the filling factor is not an integer - the qp's have fractional charge. In the case of $\nu = 1/q$ the qp can then be interpreted as $1/q$ of a missing electron or $1/q$ of an electron respectively. This treatment of qp's is strongly reminiscient of Wilczek's anyons in [16], and through other means we will see that the qp's are indeed anyonic.

The upshot is that moving the filling factor away from the middle of the plateaus produces gapped quasiparticles. These are then localized by the disorder potential in a similar way as

in the IQHE, giving the plateaus in transverse resistivity accompanied by zero longitudinal resistivity. The details of the qp production are irrelevant in this context as long as the smallest possible excitation is gapped and localized.

Inspired by the above, Laughlin's proposition [18] for a trial wavefunction describing a qh at η is

$$\Psi^L_{1/q,qh}(\eta,\{z_i\}) \equiv \prod_{i=1}^{N}(\eta - z_i)\Psi^L_q = \prod_{i=1}^{N}(\eta - z_i)\prod_{i<j}^{N}(z_i - z_j)^q e^{-\sum_i \frac{|z_i|^2}{4l^2}} \tag{4.2.6}$$

The extra factor creates a correlation hole at η so that it corresponds to fewer electrons. If we place the qh at $\eta = 0$ and consider (4.1.2) we see that we have raised the angular momentum of all the electrons by one unit, leaving an unoccupied state in the center. If we move any electron with coordinate z_i around origo the prefactor gives an additional phase of 2π. This means that the qh is a vortex; a point of zero density giving a phase factor 2π when traversed. A further confirmation of the appropriateness of $\Psi^L_{1/q,qh}$ makes use of the aforementioned plasma analogy, introduced by Laughlin [18].

4.2.1 The plasma analogy

We interpret the modulus squared of the normalized wavefunction as the Boltzmann weight in the partition function of a fictional classical two-dimensional plasma and look at the corresponding energy function:

$$\left|\Psi^L_{1/q,qh}\right|^2 = e^{-\beta U} \quad \Longrightarrow \quad U = \frac{2}{\beta}\left(-\ln|\mathcal{N}| - \sum_{i=1}^{N}\ln|\eta - z_i| - q\sum_{i<j}^{N}\ln|z_i - z_j| + \frac{1}{4l^2}\sum_{i=1}^{N}|z_i|^2\right) \tag{4.2.7}$$

To make use of this we take a look at Coloumb interactions in two dimensions. We use Gauss's law to get the potential between two point particles with charge q and relative position $\mathbf{r}$:

$$2\pi\int_A \rho_Q dA = \int_{\mathbf{l}} \mathbf{E}\cdot d\mathbf{l} = -\int_A \nabla^2 V dA \quad \Longrightarrow \quad -\nabla^2 V = 2\pi Q\delta^{(2)}(\mathbf{r}) \tag{4.2.8}$$

where A is an area bounded by the curve $\mathbf{l}$. Furthermore we have

$$\int_A \nabla^2\ln(r)dA = \int_A \nabla\cdot\left(\frac{1}{r}\partial_r(r\partial_r)\mathbf{e}_r\right)\ln(r)dA = 2\pi \quad \Longrightarrow \quad \nabla^2\ln(r) = 2\pi\delta^{(2)}(\mathbf{r}) \tag{4.2.9}$$

Then we compare this to (4.2.7). Since $|z| = r$ we see that if we choose the inverse temperature as $\beta = 2/q$ the second and third term correspond to two-dimensional Coloumb potentials, where the particles at $\{z_i\}$ have charge q and there is an impurity at η with unit charge. As for the last term, since $\nabla^2 q|z|^2/4l^2 = q/l^2$ this is can be interpreted as the potential between the particles of charge q and a uniform neutralizing background with charge density $\rho_B = -1/2\pi l^2$.

To minimize the energy of our fictional plasma we should have overall charge neutrality. This gives us the mass density of electrons n_e and thus the expected filling factor:

$$\rho_B + n_e q = 0 = -\frac{1}{2\pi l^2} + n_e q \quad \Longrightarrow \quad \nu = \frac{N}{A2\pi l^2} = \frac{n_e}{2\pi l^2} = \frac{1}{q} \tag{4.2.10}$$

since N is the number of occupied states and the number of available states in the area A is $A2\pi l^2$.

The charge of the impurity at η has the same sign as the electrons and therefore these will be repelled, effectively creating a localized entity with the opposite charge. This is what we expect for the qh and therefore confirms the appropriateness of $\Psi^L_{q,qh}$. But to maintain larger scale charge neutrality the plasma particles must screen the resulting particle. In light of their respective charges we see that $1/q$ plasma particles (which corresponds to electrons through their position vectors) are needed - and again we conclude that the qh is in a sense a fractional missing electron, bearing in mind that it is in fact an emergent entity arising from the totality of the system. Having several hints of unconventional statistics we examine a classic calculation of this.

4.2.2 Arovas, Schrieffer and Wilzcek's calculation

In their paper [21] Arovas et al (ASW) used the geometrical Berry phase to compute the statistics angle of Laughlin's quasiparticles. The Berry phase stems from the adiabatic approximation, which can be stated as follows: Assume that a system is initially in an energy eigenstate $H(\mathbf{R}_0)|\lambda(\mathbf{R}_0), t=0\rangle = E_\lambda(\mathbf{R}_0)|\lambda(\mathbf{R}_0),0\rangle$. The Hamiltonian depends on *slowly* changing parameters constituting an abstract vector $\mathbf{R}(t)$ with the starting configuration $\mathbf{R}(0)=\mathbf{R}_0$. Then at a later time the system stays in the eigenstate apart from a possible phase factor [22]:

$$|\lambda(\mathbf{R}_0),t\rangle = e^{i\gamma_\lambda(t)}|\lambda(\mathbf{R}(t)),0\rangle \tag{4.2.11}$$

This entails that if the parameters change slowly enough the system will stay in the energy eigenstate although this state changes with the parameters. What is meant by slow change above is quantitatively defined [23], and roughly means that it is very slow compared to the proper motion of the system.

One contribution to the phase γ_λ is the usual kinetic term $-\frac{i}{\hbar}\int_0^t E_\lambda(\mathbf{R}(t'))dt'$ but there is the possibility for an additional γ_B. Substituting (4.2.11) into Schrödinger's equation we obtain

$$\begin{gathered}
H(\mathbf{R}(t))|\lambda(\mathbf{R}_0),t\rangle = E_\lambda(\mathbf{R}(t))e^{-\frac{i}{\hbar}\int_0^t E_\lambda(\mathbf{R}(t'))dt'}e^{i\gamma_B}|\lambda(\mathbf{R}(t)),0\rangle = i\hbar\partial_t|\lambda(\mathbf{R}_0),t\rangle = \\
i\hbar\left(-\frac{i}{\hbar}E_\lambda(\mathbf{R}(t)) + id_t\gamma_B + d_t\right)|\lambda(\mathbf{R}(t)),0\rangle \qquad \Longrightarrow \qquad (d_t\gamma_B)|\lambda(\mathbf{R}(t))\rangle = id_t|\lambda(\mathbf{R}(t))\rangle \\
\Longrightarrow \qquad d_t\gamma_B = i\langle\lambda(\mathbf{R}(t))|d_t|\lambda(\mathbf{R}(t))\rangle
\end{gathered} \tag{4.2.12}$$

where γ_B is called Berry's phase.

Following Arovas et al we return to Laughlin's proposal for a qh wavefunction (4.2.6). We choose the qh position as our single adiabatic parameter so that $\mathbf{R}=\eta$ and assume that the qh is moved in a complete clockwise circle in the system. Thus we have

$$d_t\Psi^L_{1/q,qh} = \sum_{j=1}^N\left((d_t\eta)\prod_{i\neq j}^N(\eta-z_i)\right)\Psi^L_{1/q} = \left(\sum_{i=1}^N d_t\ln(\eta-z_i)\right)\Psi^L_{1/q,qh} \tag{4.2.13}$$

We then substitute the above into (4.2.12):

$$d_t\gamma_B = i\langle\Psi^L_{1/q,qh}|d_t\sum_{i=1}^{N}\ln(\eta-z_i)|\Psi^L_{1/q,qh}\rangle = i\langle\Psi^L_{1/q,qh}|\int d^2z\sum_{i=1}^{N}\delta(z-z_i)d_t\ln(\eta-z)|\Psi^L_{1/q,qh}\rangle \tag{4.2.14}$$

Next we define the mean electron density $\rho^{qh} \equiv \langle\sum_i \delta(z-z_i)\rangle = \rho_0 + \Delta\rho^{qh}$ where ρ_0 is the density without the qh. Assuming the density perturbation due to the qh can be neglected we get

$$\gamma_B = \int dt\left(i\int d^2z\rho_0 d_t\ln(\eta-z)\right) = i\int d^2z\rho_0\oint d\eta d_\eta\ln(\eta-z) = -i\int_{|z|\le R} d^2z\rho_0 2\pi i = 2\pi\langle N\rangle_R \tag{4.2.15}$$

where R is the radius of the qh orbit and $\langle N\rangle_R$ is the expectation value of number of electrons within. In the above we have changed integration variable from t to η and used the residue theorem, which picks out values of $|z|$ within the orbit.

By using $\langle N\rangle_R = \Phi(R)/\Phi_0$ and comparing with the Aharonov-Bohm phase we see that the qh's have fractional charge $Q_{qh} = \nu e$. Furthermore we may ask what happens if another qh resides within the loop. According to earlier arguments the qh corresponds to $1/q$ of a missing electron, and so negative change, in the phase, so that the phase change if another qh is present is

$$\Delta\gamma_B = -\frac{2\pi}{q} \tag{4.2.16}$$

And since moving one qh around another is equivalent with two particle interchanges this means that the statistics phase of the qh's should be $\theta_{qh} = -\frac{\pi}{q}$. Two comments are in order at this point:

Firstly, although the arguments above concern the Laughlin sequence the result is in fact more general. We will see that more elementary qp's than these exist at more general filling fractions, but it is always possible to create Laughlin qp's using the solenoid method and these will correspond to ν of a missing electron or of an electron respectively, cf (4.2.5). Using the same arguments the (unnormalized) qh wavefunction is $\Psi_{\nu,Lqh} = \prod_i(\eta - z_i)\Psi_\nu$ where Lqh means Laughlin quasihole, giving the same statistical exchange phase $\theta_{Lqh} = -\nu\pi$.

Secondly the magnetic field restricts the single-particle electron motion so that it can be seen as effectively one-dimensional (as long as we neglect LL mixing). This manifests itself when we restrict the motion to a single LL so that the operators α and $\alpha^\dagger$ are unused (see section 2.1) and the dynamics are described by the single harmonic oscillator algebra of β and $\beta^\dagger$ [2]. This corresponds to a one-dimensional particle in an oscillator potential, where the canonical commutation variables are the guiding center coordinates C_x and C_y rather than the usual x and p_x.

In a 1992 paper [24] Hansson et al look at the consequenses of this, but using point particle anyons rather than the extended Laughlin quasiparticles. They consider two ways of obtaining a one-dimensional system; either by restricting the dynamics to one dimension in the first place or by dimensional reduction of two-dimensional particles in a magnetic field as described in the

[2]Since the relevant particles for us is qp's rather than electrons and the former do not have LL excitations this is even more appropriate.

preceding paragraph. It turns out that when using the first approach the resulting states are labelled by a parameter θ which is the statistics angle. Furthermore they show that the system arising from dimensional reduction can equivalently be achieved by quantizing an $SP(1,\mathbb{R})$ algebra. Then when looking at interchange of coherent states of this algebra the conclusion is that the resulting Berry phases equal *the negative* of θ.

In [25] Kjønsberg and Leinaas consider the question of how this applies to Laughlin qh's rather than general anyons. By including a boundary they show that the relevant algebra is no longer that of $SP(1,\mathbb{R})$ but that of $SU(2)$, so that the coherent states are different. By expressing the wavefunction of a single qh in angular momentum eigenstates and normalizing Ψ_L^{qh} using the plasma analogy they then show how qh's are related to the $SU(2)$ coherent states. More importantly to us they conclude that the qh states evolve into the coherent states of the $SP(1,\mathbb{R})$ when the area goes to infinity. When considering two particles the relation is more complicated but the result is similar. The upshot is that apparently the phase obtained from a naive Berry phase computation is the negative of the actual statistics phase. When taking this into account the results from the ASW calculation agree with the ones from other models we will encounter later in the text. Therefore, although not rigorously proven, we conclude

$$\theta_{Lqh} = \nu\pi \tag{4.2.17}$$

4.2.3 Laughlin quasielectrons

When searching for a trial wavefunction describing qe's it is natural to look in a similar direction as for the qh. However we want a function constituting an antivortex rather than a vortex because the number of available states should be reduced. Since we want to stay in the LLL, using the complex conjugate of electron position $\overline{z}$ is not an option. Laughlin's proposition [18] is

$$\Psi_{\nu,qe} \equiv \prod_{i=1}^{N}(\overline{\eta} - 2l^2\partial_{z_i})\Psi_\nu = \prod_{i=1}^{N}(\overline{\eta} - 2l^2\partial_{z_i}) \prod_{i<j}^{N}(z_i - z_j)^q e^{-\sum_i \frac{|z_i|^2}{4l^2}} \tag{4.2.18}$$

where the differential operator only acts on the polynomial part of the wavefunction. If we choose $\overline{\eta} = 0$ we see that all the relative angular momenta are reduced by one unit. This entails shrinking the area but keeping the number of electrons constant so that the density is slightly higher than without the qe, as expected. Another way of justifying the differential is to start with $\overline{z}$ and project the result to the LLL:

Following [8] we consider a general operator $\mathcal{O}(z,\overline{z})$. We demand that the projected operator acting on states in the LLL should yield states also in the LLL, and that the matrix elements between states in the LLL of the original and projected operator are the same. Then we have

$$\mathcal{P}_{LLL}\mathcal{O} = e^{-\frac{z\overline{z}}{4l^2}} : \mathcal{O}(\overline{z} \to 2l^2\partial_z, z) : e^{\frac{z\overline{z}}{4l^2}} \tag{4.2.19}$$

where the symbol : signifies moving all instances of $\overline{z}$ to the left of all z's. This is equivalent to removing the exponentials and :'s and letting the differentials act only on polynomials. To prove the above we expand the operators as $\mathcal{O} = \sum_{j.k} C_{jk}\overline{z}^j z^k$ and recall that $|0\ m\rangle \propto$

$z^m \exp(-z\overline{z}/4l^2)$. This means that

$$\begin{aligned}
\langle 0\ m|\mathcal{O}|0\ p\rangle &= \overline{N}_{0m}N_{0p}\sum_{j,k}C_{jk}\int d^2z\overline{z}^m e^{-\frac{z\overline{z}}{4l^2}}(\overline{z}^j z^k)z^p e^{-\frac{z\overline{z}}{4l^2}} \ \sim \\
\int d^2z\overline{z}^m z^{k+p}(-2l^2\partial_z)^j e^{-\frac{z\overline{z}}{2l^2}} &= (-2l^2)^j\left(\left[\overline{z}^m z^{k+p}\partial_z^{j-1}e^{-\frac{z\overline{z}}{2l^2}}\right]_\Omega - \int d^2z(\partial_z\overline{z}^m z^{k+p})\partial_z^{j-1}e^{-\frac{z\overline{z}}{2l^2}}\right) \\
&= \ldots = \int d^z e^{-\frac{z\overline{z}}{2l^2}}\overline{z}^m(2l^2\partial_z)^j z^{k+p} \ \sim \\
\overline{N}_{0m}N_{0p}\sum_{j,k}C_{jk}\int d^2z\overline{z}^m e^{-\frac{z\overline{z}}{4l^2}}&\left(e^{-\frac{z\overline{z}}{4l^2}}(2l^2\partial_z)^j z^k e^{\frac{z\overline{z}}{4l^2}}\right)z^p e^{-\frac{z\overline{z}}{4l^2}} = \langle 0\ m|\mathcal{P}_{LLL}\mathcal{O}|0\ p\rangle
\end{aligned} \tag{4.2.20}$$

where the symbol $\sim$ indicates neglecting to write normalization factors, summation symbol and expansion coefficients. We have used repeated partial integration and assumed that the boundary terms at Ω disappear. With that (4.2.19) is proven, and we may obtain Laughlin's qe wavefunction by projecting to the LLL the complex conjugate of the qh wavefuntion.

$\Psi^L_{1/q,qe}$ is however more difficult to handle than its qh counterpart. According to ASW [21] the charge is $-e/3$. Furthermore, using the same logic as in section 4.2.2 and that the expectation value of number of electrons within the circle should increase as the flling factor when a qe is inside, we get the following relative statistics angle between a qh and a qe:

$$\theta_{Lqh\leftrightarrow Lqe} = -\nu\pi \tag{4.2.21}$$

Next we introduce an argumentation technique we will refer to as clustering arguments, used early by Su in [26] and which we will utilize more later in the text. In this we imagine interchanging composite objects, or clusters of particles. We do not provide a detailed microscopic description of the resulting states, but simply argue that if two particles A and B are interchanged with two particles C and D the resulting phase factor on the wavefunction should logically be

$$\theta_{AB\leftrightarrow CD} = \theta_{A\leftrightarrow C} + \theta_{A\leftrightarrow D} + \theta_{B\leftrightarrow C} + \theta_{B\leftrightarrow D} \tag{4.2.22}$$

and similar for more particles. This is justified by locality in [27]. Another useful observation is that since the configuration space singularity lies only in the *relative* space the effect of interchanging a particle A around a particle B is the same as the opposite (as long as they are interchanged in the same sense; our default is clockwise). Thus we have

$$\theta_{A\leftrightarrow B} = \theta_{B\leftrightarrow A} \tag{4.2.23}$$

To find the statistics of the qe's we imagine interchanging two identical composites consisting of one qh and one qe each, see figure 4.3. As long as the clusters do not move too close to each other this is a trivial operation, since seen from a distance a qh and a qe equals the vacuum. With this in mind and using (4.2.21), (4.2.22) and (4.2.23) we have

$$\begin{aligned}
\theta_{Lqh,Lqe\leftrightarrow Lqh,Lqe} = 0 &= \theta_{Lqh} + 2\theta_{Lqh\leftrightarrow Lqe} + \theta_{Lqe} \\
\implies \qquad \theta_{Lqe} &= \nu\pi
\end{aligned} \tag{4.2.24}$$

so that the statistics angle is expected to be the same for Laughlin qh's and qe's.

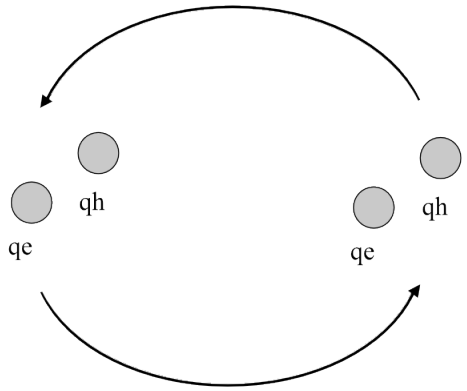

Figure 4.3: Exchanging two composites consisting of a quasielectron and a quasihole

In [28] Kjønsberg and Myrheim investigated numerically the Berry phases of $\Psi^L_{1/3,qh}$ and $\Psi^L_{1/3,qe}$. The results for electric charges were as expected: $e/3$ and $-e/3$ respectively with finite-size and boundary corrections. The statistics of the qh gave no surprises neither; away from the boundary and with particle separations not too small to avoid overlap the result was $\theta_{qh} = \pi/3$. However the qe statistics angle did not converge towards a single value even with 200 particles. The interpretation was that Laughlin's trial wavefunction for quasielectrons, although having some correct properties, was ill-fitting and that we need another prescription.

A recent article by Jeon and Jain [29] complicates this picture. They revisit the numerical calculation of the Laughlin qe statistics angle and show that in the thermodynamic limit the expected result is recovered[3], and also that two seemingly equivalent computation methods behave differently in this respect. The two methods entail calculating the effects on the wavefunction of direct exchange of particles (as in Kjønsberg and Myrheim's article) and computing the change in Berry phase of a particle moving in a loop when another inside the trajectory (similar in spirit to Arovas et al's article), respectively.

While the first method proves to be very sensitive to the particle number N (explaining the result in [28]) the second is not - the expected angle is extracted with 200 and fewer electrons as long as the exchange path is small compared to system size. Jain and Jeon argue that the reason is that Kjønsberg and Myrheim's method relies on calculating Aharonov-Bohm phases. While we have implicitly assumed that the particle charge in these phases are the same as the local charge of the qp's relative to the electron liquid vacuum, they show that ϕ_{AB} is affected by an area and N-dependent term (as also shown in [28]) while the Berry phase is not. Thus, if the number of particles is too small and/or the area of the loop is too big the answer is compromised (charge calculations are unaffected because of a smaller area). We assume that these considerations affect numerical calculations only.

[3]They use different conventions where the expected angles are $\theta = -\pi/3$.

4.3 The hierarchy

Laughlin's model of systems at filling factors $\nu = 1/q$ automatically translates into a description of filling factors $\nu = 1 - 1/q$ by use of the QHE system's exact particle-hole symmetry. As shown in [9] the Hamiltonian is unchanged apart from an unimportant constant when expressed in hole-operators instead of electrons. If we use creation/annihilation operators $\epsilon^\dagger_{m'}, \epsilon_{m'}$ and $\kappa^\dagger_{m'}, \kappa_{m'}$ for electrons and holes respectively with angular momentum m' (all in the LLL) their vacua are related as

$$|0\rangle_\epsilon = \sum_{m'} \kappa^\dagger_{m'}|0\rangle_\kappa \quad \text{and} \quad |0\rangle_\kappa = \sum_{m'} \epsilon^\dagger_{m'}|0\rangle_\epsilon \tag{4.3.1}$$

so that the electron vacuum corresponds to a full LLL for holes and vice versa. Thus the spectrum at $\nu = 1 - 1/q$ is identical to the one at $\nu = 1/q$ apart from the aforementioned constant (which can be traced to interactions between the holes and heir vacuum [9]), and our explanation for the Laughlin sequence is extended to these filling factors. However the QHE has also been observed at fractions other than these.

An early scheme devised to describe the remaining QHE fillings, introduced by Haldane [30], Halperin [31] and others, is referred to as the hierarchy construction. Although conceptually successful and predicting topological quantities this scheme ends up with complicated integrals and it is difficult or impossible to obtain quantitative results. We will not go in detail on the hierarchy model but describe the essentials and how to predict statistical angles of the qp's.

Instead of shifting focus and using holes as the dynamic quantities, as we did in the beginning of this section, we use qp's. We start out with a stable state on a Laughlin plateau, that is a filling factor of $\nu = 1/q$. Then we increase or decrease the magnetic field so as to create qh's or qe's respectively. These particles are local entitities repelling each other electrically much like the original electrons even though they have fractional charge and statistics[4]. When reaching a certain number of qp's the hierarchic model states that they will condense into a correlated liquid with a mobility gap and thus manufacture a new plateau in the resistivity, in an analogous way to the electrons in the Laughlin state. Changing the magnetic field again we may then create qh's or qe's in this new ground state, different from the ones arising from the original Laughlin state, and these may again condense into a gapped liquid. Thus we have a hierarchy of states, starting from somewhere on the Laughlin sequence and then reaching more exotic QHE filling fractions.

The idea is then to change representation from an electron wavefunction Ψ to a qp pseudowavefunction Φ. We denote the hierarchy level with n, so that $n = 1$ implies the Laughlin sequence. The electron wavefunction is [32]

$$\Psi^H_{\nu_{n+1}}(\{z_i\}) = \prod_{i=1}^{M_n} \int d^2\eta_i \Phi^*_{n+1}(\eta_1, \ldots, \eta_{M_n}) \Psi^H_{\nu_n, Mqp}(\eta_1, \ldots, \eta_{M_n}, z_1, \ldots, z_N) \tag{4.3.2}$$

where η_i are the coordinates of the M_n qp's required, z_i are the coordinates of the N electrons and $\Psi_{\nu_n, Mqp}$ is the wavefunction of the system at level n with M qp's (when the condensing particles are qe's we have $\eta \to \overline{\eta}$). Thus $\Psi^H_{\nu_1}$ are the familiar qp Lauglin states we have seen

[4] The similarity of electrons and qp's is examined in [**??**].

in (4.2.6) and (4.2.18). In [33] Laughlin shows that in a limiting case the above procedure constitutes particle-hole conjugation.

In [34] Halperin argues that the pseudowavefunction may be written on the form

$$\Phi_{n+1}(\{\eta_i\}) = \prod_{j<k}^{M_n}(\eta_j - \eta_k)^{2p_{n+1}-\alpha_{n+1}/s_n} e^{-\sum_l \frac{\zeta_n|\eta_l|^2}{4l^2}} \tag{4.3.3}$$

where s and p are integers characterizing the state. p can be choosen at will to describe states of different density, and he defines $s_{n+1} = 2p_{n+1} - \alpha_{n+1}/s_n$. α determines whether the state is created by condensing qh's or qe's so that the particles condensing to form level $n+1$ have electronic charge $Q = -\alpha_{n+1}\zeta_n e$.

We see that Φ resembles Laughlin wavefunctions, which is reasonable since it is supposed to describe a similar system, only with different particles. The differences are the qp charge in the exponential and the possibility of a fractional exponent on the polynomial. If this possibility is fulfilled the wavefunction is manifestly multivalued, since interchanging two qp's will lead to a complex factor on the wavefunction itself. This is called monodromy. Remembering that when describing qe's we should substitute $\overline{\eta}$ for η we see that the statistics angles arising from the monodromy are

$$\theta_{qh} = \theta_{qe} = \pi/s_n \tag{4.3.4}$$

In section 6.2.1 we consider a similar wavefunction also with monodromy and show that it has no contribution from exchange statistics to the Berry phase. We assume that the same holds here so that the total statistics angle of Φ can be read out from the monodromy. Due to the form of (4.3.2) it is not clear that this is the actual statistics of the qp's, but this is made probable since $|\Phi|^2$ gives the probability distribution of qp's [35] in a similar way as $|\Psi|^2$ for the electrons. In addition the results agree with the other models, as seen in section 7.4.

To find iterative expressions for the pseudowavefunction parameters Halperin uses the plasma analogy with Φ. Choosing inverse temperature $\beta = 2/s_{n+1}$ the resulting plasma potential is

$$U = -s_{n+1}^2 \sum_{j<k}^{M} \ln|\eta_j - \eta_k| + s_{n+1}\sum_{l=1}^{M}\frac{|\zeta_n||\eta_l|^2}{4l^2} \tag{4.3.5}$$

In analogy with section 4.2.1 we recognize this as the potential of interacting particles with charge s_{n+1} embedded in a background potential of charge density $-|\zeta_n|/2\pi l^2$. To maintain charge neutrality the mass density of qp's must be $|\zeta_n|/s_{n+1}$ (cf. (4.2.10)). From the condensing qp charge $Q = -\alpha_{n+1}\zeta_n e$ we conclude that the corresponding electron filling fraction is

$$\nu_{n+1} = \nu_n + \Delta\nu = \nu_n + \frac{\alpha_{n+1}|\zeta_n|\zeta_n}{s_{n+1}} \tag{4.3.6}$$

In the same way as for the Laughlin case we can create "quasi-quasiparticles" in the pseudowavefunction by factoring in vortices and antivortices:

$$\Phi_{n+1,qh}(\xi,\{\eta_i\}) = \prod_{j=1}^{M_n}(\xi - \eta_j)\Phi_{n+1} \qquad \Phi_{n+1,qe}(\overline{\xi},\{\eta_i\}) = \prod_{j=1}^{M_n}(\overline{\xi} - 2\partial_{\eta_j})\Phi_{n+1} \tag{4.3.7}$$

and from (4.3.3) we see that these qp's correspond to $1/s_{n+1}$ of a missing electron or of an electron respectively, yielding the following expression for the charge factor:

$$\zeta_{n+1} = \frac{\alpha_{n+1}\zeta_n}{s_{n+1}} \tag{4.3.8}$$

This is consistent with (4.3.6) since α fixes what kind of condensate we get at the next level (and the physical electric charge has the opposite sign because ν counts electrons). With the initial configuration $\zeta_0 = s_0 = \alpha_1 = 1$ and $\nu_0 = 0$ the known expressions are reproduced, and we may construct various QHE systems with different sequences $\{\alpha_s, p_s\}$.

If we choose $p_1 = 1$ the level one system has $\nu_1 = 1$; a completely full LLL. The qp charges are then $Q = \pm e$; corresponding to electrons and holes. Building an $n = 2$ system from this starting point we find filling fractions $\nu_2 = 2, 4/3, 6/5, 8/7\ldots$ and $\nu_2 = 2/3, 4/5, 6/7\ldots$ The former list of fractions is the Laughlin series in the second LL, consistent with the condensing qp charges $Q = -e$ representing electrons. The latter list is reached using holes, $Q = e$, and constitutes the particle-hole conjugate Laughlin sequence $\nu = 1 - 1/q$.

With $p_1 = 2, 3, \ldots$ the $n = 1$ fillings are the Laughlin fractions $\nu_1 = 1/3, 1/5, \ldots$ with the expected qp charges $Q = \pm e/3, \pm e/5, \ldots$ From these we can manufacture QHE systems with Jain sequence filling fractions and others. For example, starting with $\nu_1 = 1/3$, we may condense qh's and obtain the lower fractions $\nu_2 = 2/7, 4/13, \ldots$ or condense qe's and get the higher fillings $\nu_2 = 2/5, 4/11, \ldots$

The iterative equations (4.3.4), (4.3.6) and (4.3.8) then enable us to calculate the qp statistics angle of any filling fraction on the hierarchy. However we will see that there may exist different qh's at a given filling factor, something this construction does not say anything about. We will return to this point later. As shown in [30] the filling factors can be written

$$\nu_n = \cfrac{1}{2p_1 - \alpha_1 - \cfrac{\alpha_2}{2p_2 - \cfrac{\alpha_3}{\ddots \cfrac{}{2p_{n-1} - \cfrac{\alpha_n}{2p_n}}}}} \tag{4.3.9}$$

In the same article Halperin argues that the mobility gap, and thus the stability of the QHE state, is determined by the denominator of ν with weaker states for higher numbers. This is in concurrence with experiments and in opposition to the sometimes stated view (e.g. in [8]) that hierarchical states should have lower stability with higher n, which is contradicted by experiment. However, as mentioned the integrals resulting at the end of the day are intractable. In the next chapter we will look at another scheme which does enable us to write down explicit electron wavefunctions.

At this point we choose which filling factors to focus on. Since the stability falls with increasing denominator we include no $\nu = p/q$ where $q > 19$. Moreover we will see in chapters 5 and 6 that the states emerging from qh rather than qe condensates are harder to describe using the composite fermion and conformal field theory approaches; since our aim is to compare the models we omit such states by choosing $\alpha_i = 1\ \forall\ i$. Lastly we stop at level $n = 6$ and include no initial Laughlin state which does not yield a daughter state following these criteria. A schematic representation of the filling factors we end up with is displayed in figure 4.4. Most of these states have been observed in experiment.

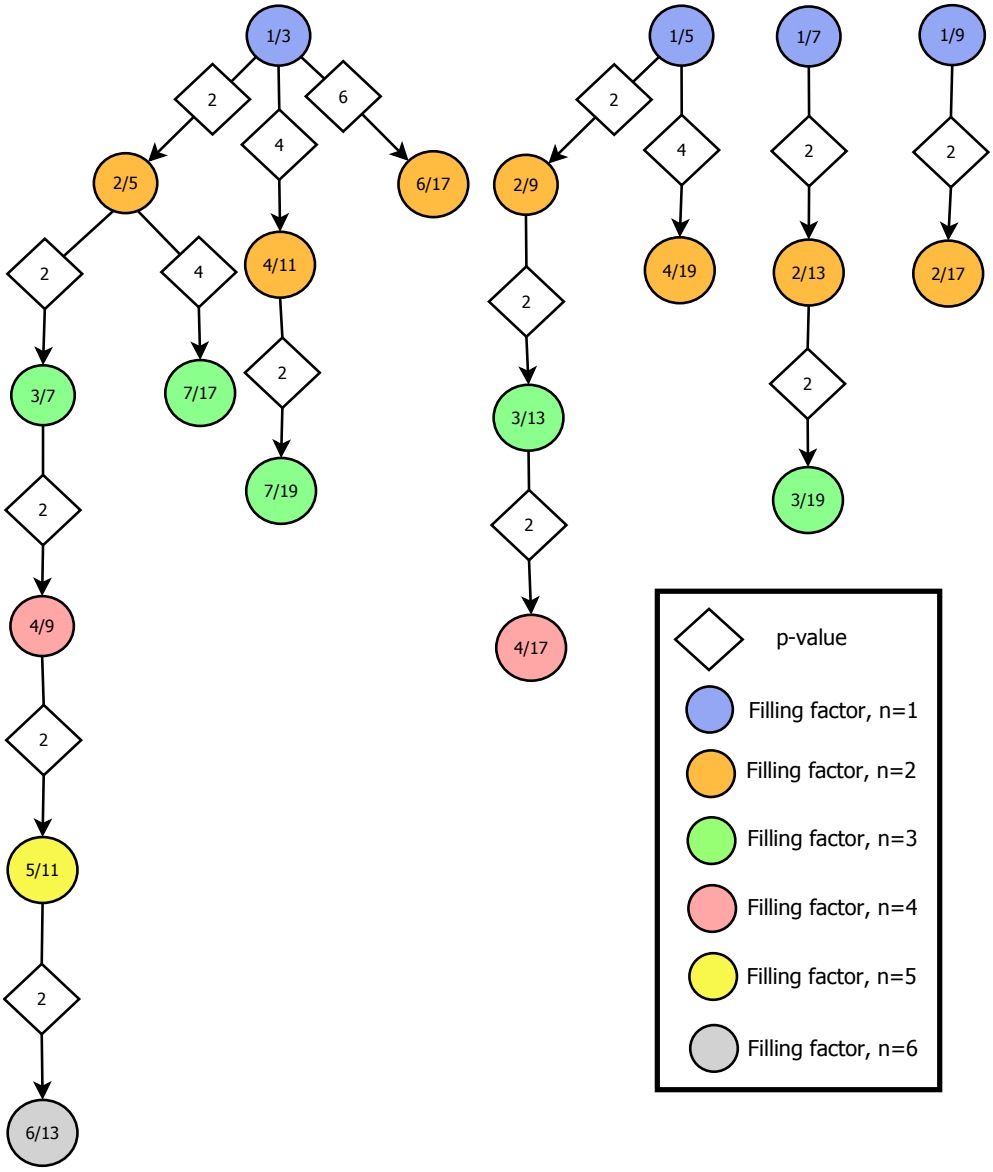

Figure 4.4: The selected hierarchical filling fractions

Chapter 5

Composite Fermions

A different description of the FQHE due to Jain [36] is through composite fermions (CF's). The following basic introduction is based on [8][1].

5.1 Flux attachment

In this model the interacting objects are composite objects consisting of electrons bound to an even number $2p$ of magnetic flux quanta (similar schemes with an odd number, which leads to composite bosons, are also in use). Using this picture one may construct trial wavefunctions for a large class of FQHE which turn out to have very good overlap. The flux attachment can be done by multiplying a wavefunction describing electrons at positions $\{z_i\}$ with a factor

$$\prod_{j<k}^{N}(z_j - z_k)^{2p} \tag{5.1.1}$$

To see that this binds flux to the particles we picture moving the electron at z_1 completely around the one at z_2 and introduce the relative coordinates $r_{12}e^{i\phi_{12}} \equiv z_1 - z_2$. The effect on the wavefunction is then

$$\prod_{i<j}^{N}(z_i - z_j)^{2p}\Psi = r_{12}^{2p}e^{i2p\phi_{12}}\prod_{i<j}^{ij\neq 12}(z_i - z_j)^{2p}\Psi \xrightarrow{\phi_{12}\to\phi_{12}+2\pi} e^{i2p2\pi}\prod_{i<j}^{N}(z_i - z_j)^{2p}\Psi' \tag{5.1.2}$$

The prefactor gives a factor corresponding to an Aharonov-Bohm phase (see section 4.2) from $2p$ magnetic flux quanta $\Phi_0 = h/e$ on the wavefunction. Thus each CF sees $2p$ flux quanta attached to each of the others.

The next step is to use a mean field-approximation where the point fluxes are smeared out and continuous. Then we may view the CF's as objects moving in an altered magnetic field B^*, resulting from the external one minus the point fluxes the CF' see on each other. The magnetic field seen by the electrons is thus $B = B^* + 2p\rho\Phi_0$ where B^* may point in either direction and ρ is the electron density. The $\nu = 1/3$ state may for example be viewed as CF's with two point fluxes each moving in a magnetic field with one flux quantum per particle - that is, with an effective filling factor $\nu^* = 1$. The crucial assumption of the CF model is

[1] Jain uses the convention $z = x - iy$ and a magnetic field $\mathbf{B} = B\mathbf{e}_z$, giving some expressions the opposite sign from this text.

then that we will get an IQHE in the new magnetic field when the CF's occupy full LL's of B^*, giving the CF filling fraction $\nu^* = k \ ; \ k \in \mathbb{N}$. Then we may calculate the electron filling fraction:

$$\begin{aligned} \nu^* = k = \frac{\rho\Phi_0}{|B^*|} \quad &\Longrightarrow \quad |B^*| = \pm\frac{\rho\Phi_0}{k} \\ \nu = \frac{\rho\Phi_0}{B} = \frac{\rho\Phi_0}{B^* + 2p\rho\Phi_0} &= \frac{k}{2pk \pm 1} \end{aligned} \tag{5.1.3}$$

where the direction of B^* fixes the $\pm$. In addition to the Laughlin fractions obtained with $k = 1$ these ν's generate many of the observed FQHE filling fractions and are collectively called the positive or negative Jain series depending on the sign in the denominator. The negative Jain series is more difficult to handle, both in the CF model and using conformal field theory (chapter 6). Therefore we will limit ourselves to the positive sequence in the remainder.

Methods for describing the fractions not belonging to the Jain series using FQHE or pairing of CF's have been proposed but we will not consider these. In addition we may anticipate special circumstances where the CF magnetic field is zero, that is $\nu = \frac{1}{2p}$ - and true enough a state resembling a fermi sea of CF's is seen at these points numerically and in experiment. This is indeed what is expected for fermions with no magnetic field.

Encircling one CF with another gives a factor $\exp(i2p2\pi)$, which together with the fact that the wavefunction goes to zero when two CF's are at the same position means that we can also see the model as attaching $2p$ vortices to each electron. This builds repulsive correlations between them as can for example be seen from the fact that the factors $(z_i - z_j)^{2p}$ raises the angular momentu and keeps them further apart. Electrons being repulsive, it then makes sense that their interactions can be modelled by attaching vortices.

In this way the CF model describes a complex non-local phenomenon using emergent weakly interacting local objects (the most surprising thing is how good the aforementioned mean field approximation is). The CF's have the same charge as an electron and are fermions, since the factor (5.1.1) is symmetric in relative coordinates whereas the electron wavefunction is antisymmetric.

The wavefunction is a slater determinant Ψ_k describing k occupied CF LL's with an extra factor of (5.1.1). But a problem with this is the factors of $\overline{z}$ in Ψ_k; it resides partially in higher LL's even when we are supposed to be in the LLL. Therefore we use a LLL projection, yielding the CF wave functions for $\nu = \frac{k}{2pk+1}$:

$$\Psi_\nu^{cf} = \mathcal{P}_{LLL} \prod_{j<k} (z_j - z_k)^{2p} \Psi_k \tag{5.1.4}$$

where Ψ_k is evaluated at the CF magnetic field B^*. This will make sure that the particle positions are the same after flux attachment; the factors of $(z_j - z_k)$ push them outwards but the altered magnetic field in the exponents pulls them inward [37].

From section 4.2.3 we remember that the projection entails switching $\overline{z} \to 2\partial_z$ where the differentials only work on polynomials. The hope is then that the projection does not destroy the properties described above. This is made plausible by the fact that most of the wave function is already in the LLL, and corroborated by the significant success of the CF scheme, eg. in numerical overlap with the exact function. Thus we have an effective way of creating trial wavefunctions for the Jain sequence, even though the actual reality of the CF's is controversial.

5.2 CF quasiparticles

Continuing the analogy with the IQHE we assume the CF's to be localized by the disorder potential when we vary ν, giving a model for the FQHE plateaus. We see that if we lower B and move away from the Jain series, entailing creation of qe's cf. previous sections, some CF's have to occupy higher CF LL's, analogous with electrons at $\nu = k$ having to move to higher LL's. Following this logic a qe can be modelled as a CF raised to the next level (see figure **??**; a CF in a still higher LL would be an excitation) and a qh as one missing CF from a CF LL. This means that at level k there are potentially k different qh's, while the qe is unique.

On the Laughlin sequence one may show that the qh's are equal to Laughlin's proposal but not the qe's. Numerical calculations show that the energies of the CF qe's are lower than Laughlin's [38] so that they are potentially more fundamental (the Laughlin qe resembles an excited CF qe, elevated to the third CF LL). As an example we construct the wave function of a single qe at filling factor $\nu = 2/5$.

To get this filling we can use $k = 2$, $p = 1$ and a CF magnetic field pointing in the negative z-direction (that is, the same as the unaltered field so that we have a plus in the filling factor denominator). This corresponds to two completely filled CF LL's of electrons attached to two flux quanta each. We need $\Phi_{k=2}$, which as seen in section 2.1 is a slater determinant of single particle wavefunctions (without normalization)

$$\phi_{m0} \sim z^m e^{-|z|^2/4l^2} \qquad \text{and} \qquad \phi_{m1} \sim -(2m+2-z\overline{z})z^m e^{-|z|^2/4l^2} \tag{5.2.1}$$

In the slater determinant each term is a product of one of the elements of each row in an antisymmetric combination, with all possible combinations realized. With half the particles to residing in $k = 1$ and half in $k = 2$, we get (remembering that $m = -k, -k+1, \ldots$)

$$\begin{aligned}
\Psi_2 &= \begin{vmatrix}
1 & 1 & \cdots & 1 \\
z_1 & z_2 & & z_N \\
\vdots & & & \vdots \\
z_1^{N/2-1} & z_2^{N/2-1} & & z_N^{N/2-1} \\
\overline{z}_1 & \overline{z}_2 & & \overline{z}_N \\
-2+z_1\overline{z}_1 & -2+z_2\overline{z}_2 & & -2+z_N\overline{z}_N \\
(-4+z_1\overline{z}_1)z_1 & (-4+z_2\overline{z}_2)z_2 & & (-4+z_N\overline{z}_N)z_N \\
\vdots & & & \vdots \\
(-2(N-1)-2+z_1\overline{z}_1)z_1^{N/2} & & \cdots & (-2(N-1)-2+z_N\overline{z}_N)z_N^{N/2}
\end{vmatrix} e^{-\frac{1}{4l^2}\sum_i |z_i|^2} \\
&= \begin{vmatrix}
1 & 1 & \cdots & 1 \\
z_1 & z_2 & & z_N \\
\vdots & & & \vdots \\
z_1^{N/2-1} & z_2^{N/2-1} & & z_N^{N/2-1} \\
\overline{z}_1 & \overline{z}_2 & \cdots & \overline{z}_N \\
\overline{z}_1 z_1 & \overline{z}_2 z_2 & & \overline{z}_N z_N \\
\vdots & & & \vdots \\
\overline{z}_1 z_1^{N/2-1} & \overline{z}_2 z_2^{N/2-1} & \cdots & \overline{z}_N z_N^{N/2-1}
\end{vmatrix} e^{-\frac{1}{4l^2}\sum_i |z_i|^2}
\end{aligned} \tag{5.2.2}$$

where in the last step we have used the fact that adding a multiple of one row to another does not change the determinant, enabling the z^s-term of each element in the second CF LL to be removed using a multiple of row $s-1$ in the first LL.

Since we want one extra CF in the third CF LL we must place an extra electron in the third LL before attaching vortices and projecting. With an angular momentum $j=-2$ the single particle-state is

$$\phi_{m2} \sim \overline{z}^2 e^{-\frac{|z|^2}{4l^2}} \tag{5.2.3}$$

giving, with the convention that the number of electrons is now $N+1$:

$$\Psi_{2,1e} = \begin{vmatrix} \overline{z}_1^2 & \overline{z}_2^2 & \cdots & \overline{z}_{N+1}^2 \\ 1 & 1 & & 1 \\ \vdots & & & \vdots \\ z_1^{N/2-1} & z_2^{N/2-1} & & z_{N+1}^{N/2-1} \\ \overline{z}_1 & \overline{z}_2 & \cdots & \overline{z}_{N+1} \\ \vdots & & & \vdots \\ \overline{z}_1 z_1^{N/2-1} & \overline{z}_2 z_2^{N/2-1} & \cdots & \overline{z}_{N+1} z_{N+1}^{N/2-1} \end{vmatrix} e^{-\frac{1}{4}\sum_i |z_i|^2} \tag{5.2.4}$$

With that the only thing remaining is projection and we have our wave function $\Psi^{cf}_{2/5,1qe} = \mathcal{P}_{LLL}\prod_{j<k}(z_j - z_k)^2 \Psi_{2,1e}$. This qe has no well defined position, but it is an angular momentum eigenstate. To localize the qe we must use a superposition; more on this later.

A difference between Laughlin's qh's and those arising in the CF for a more general ν is that the former are true vortices where the wavefunction goes to zero while the latter are not. However we may always create such a vortex by placing one qh at each CF LL. In figure 5.1 is a graphic representation of this for a Lqh with the lowest angular momentum; we picture moving all the CF's one step to the right or in other words increasing all the angular momenta with one unit. This has the following effect on the integer electron filling factor part of the wavefunction:

$$\Psi_k = \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ z_1^{N/k-1} & \cdots & z_N^{N/k-1} \\ \overline{z}_1 & \cdots & \overline{z}_N \\ \vdots & & \vdots \\ \overline{z}_1 z_1^{N/k-1} & \cdots & \overline{z}_N z_N^{N/k-1} \\ \vdots & & \vdots \end{vmatrix} e^{-\frac{1}{4l^2}\sum_{i=1}^N |z_i|^2} \longrightarrow \begin{vmatrix} z_1 & \cdots & z_N \\ \vdots & & \vdots \\ z_1^{N/k} & \cdots & z_N^{N/k} \\ \overline{z}_1 z_1 & \cdots & \overline{z}_N z_1 \\ \vdots & & \vdots \\ \overline{z}_1 z_1^{N/k} & \cdots & \overline{z}_N z_N^{N/k} \\ \vdots & & \vdots \end{vmatrix} e^{-\frac{1}{4l^2}\sum_{i=1}^N |z_i|^2}$$

$$= \prod_i^N z_i \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ z_1^{N/k-1} & \cdots & z_N^{N/k-1} \\ \overline{z}_1 & \cdots & \overline{z}_N \\ \vdots & & \vdots \\ \overline{z}_1 z_1^{N/k-1} & \cdots & \overline{z}_N z_N^{N/k-1} \\ \vdots & & \vdots \end{vmatrix} e^{-\frac{1}{4l^2}\sum_{i=1}^N |z_i|^2} = \prod_i^N z_i \Psi_k \tag{5.2.5}$$

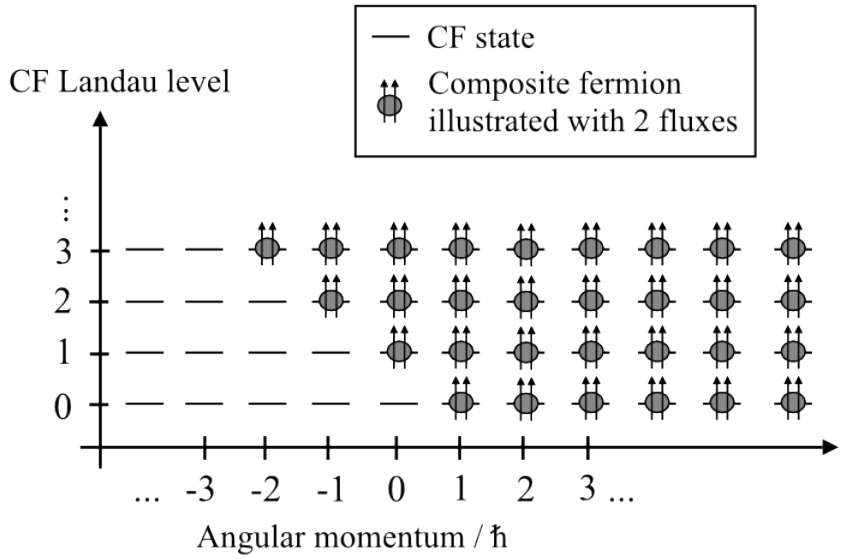

Figure 5.1: Laughlin qh. In each CF LL the lowest angular momentum state (equal to $-k$) is empty.

so that the complete wavefunction is transformed as $\Psi \to \prod_i z_i \Psi$. This then creates a vortex, a Laughlin qh, at the origin.

The aforementioned method of LLL projection is a big task and numerically untractable. Therefore the following projection method created by Jain is normally used instead: First we observe that

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{12} & a_{22} & \dots & \vdots \\ \vdots & & & \\ a_{N1} & & \dots & a_{NN} \end{vmatrix} \prod_{i<j} (z_i - z_j)^{2p} = \begin{vmatrix} a_{11} \prod_{k\neq 1}(z_1 - z_k)^{2p} & \dots & a_{1N} \prod_{k\neq N}(z_N - z_k)^{2p} \\ \vdots & & \vdots \\ a_{N1} \prod_{k\neq 1}(z_N - z_k)^{2p} & \dots & a_{NN} \prod_{k\neq N}(z_N - z_k)^{2p} \end{vmatrix} \tag{5.2.6}$$

Then we use this and take $\{a_{ij}\}$ as the elements of the slater determinant $\Psi_{2,1e}$, replace the $\overline{z}$'s in the resulting elements with derivatives and move them to the left before performing the differentiation *within* each element. This is not equivalent with the former projection method of course, because the derivatives now act on much fewer z's. But it turns out that they, in the cases where both can be calculated, give almost the same results. Besides there is no way of determining a priori how we should project the wavefunctions, and the method described above is effective and gives good overlap.

5.2.1 Charge and statistics

At first glance it is difficult to reconciliate the above quasiparticles, which are CF's with unit charge and fermionic statistics, with the ones encountered in the Laughlin and hierarchical wavefunctions. And because general arguments relying on incompressibility at fractional fillings and gauge invariance alone predict fractional charge and statistics it might seem that the

CF theory is at fault. Investigating the CF wavefunctions numerically however, one finds that the quasiparticles do indeed exhibit fractional charge and statistics [37, 39]. This means that these are in some sense hidden. Jain promotes qualitative arguments to show this analytically in arguing that it is a matter of point of view: There are two choices for what we may call the vacuum - with reference to the true vacuum the CF's are similar to electrons, but another choice is to view the incompressible electron fluid at the nearest Jain state ν as the ground state.

There are several ways to find the quasiparticle charge compared to the latter vacuum. The vortices create correlation holes as can be seen directly from the wavefunctions' extra zeros. The qp charge associated with the alternative vacuum is then the difference between the actual charge and what would have been there in the electron fluid without the qp. When a CF is moved in a circle in the fluid, it picks up a phase 2π for each vortex in addition to the normal Aharonov-Bohm phase, as seen from the factor $(z_i - z_V)$ where z_V is the vortex position. If we attribute this extra phase as the Aharonov-Bohm phase of the vortex we get (similar to the discussion in section 4.2.2)

$$2\pi = \frac{2\pi}{\nu}\frac{Q_V}{e} \quad \Longrightarrow \quad Q_V = \nu e \tag{5.2.7}$$

and since each CF consists of one electron and $2p$ vortices the total charge relative to the electron fluid vacuum is

$$Q_{CF} = -e + 2p\nu e = e\left(\frac{2pn}{2pn+1} - 1\right) = -\frac{e}{2pn+1} \tag{5.2.8}$$

for B^* pointing in the negative direction. This is the charge of a CF qe, while the CF qh charge, being an abscence of a CF, has the same value with the opposite sign. Because of the lower charge these are more fundamental than the vortices and antivortices constituting Laughlin qp's.

A thorough analytical calculation of the CF qp statistics angles through Berry phases has not been performed (although as we will see numerical ones exist). But following Jain we have somewhat heuristic arguments stating explicit expressions for the qe angles:

We imagine moving one qe around another in a circle of radius R in the QHE system. Through the flux attachment the wavefunction has factors $(z_i - z_j)^{2p}$, and with an equivalent calculation as in section 4.2.2 one may show that the difference due to the second qe in the Berry phase aquired after the exchange is $\Delta\gamma_B = 2p2\pi\Delta\langle N\rangle_R$. From the charge we can find the change in the average number of electrons and thus in the Berry phase:

$$\Delta\langle N\rangle_R = \frac{Q_{qe}}{e} = \frac{1}{2pk+1} \qquad \Longrightarrow \qquad \Delta\gamma_B = \frac{2p2\pi}{2pk+1} \tag{5.2.9}$$

This is twice the change in the Berry phase due to the exchange of the vortices, because the process we used entails two interchanges. Moreover we expect the associated statistics angle to have the opposite sign, in keeping with the comments at the end of section 4.2.2. But since the qe also consists of a fermionic electron we expect an additional phase. To investigate this we need a wavefunction with two localized CF qe's. As in [37] we use the following (unnormalized) coherent superpositions for a single particle in a magnetic field:

$$\phi_\eta^k(z) \equiv (\overline{\eta} - \overline{z})^k e^{\frac{2\overline{\eta}z - |\eta|^2 - |z|^2}{4l^{*2}}} \tag{5.2.10}$$

where k is the LL, η is the qe position and the asterisk on l^* means that it is evaluated at the effective CF magnetic field. From here on we omit the exponential factors as before. The wavefunction must be antisymmetric in electron coordinates and therefore we place the single particle superpositions inside the Slater determinant:

$$\Psi^{cf}_{\nu,2qe} = \mathcal{P}_{LLL}\Psi_{k,2e}\prod_{i<j}^{N}(z_i - z_j)^{2p} = \mathcal{P}_{LLL}\begin{vmatrix} \phi^k_{\eta_1}(z_1) & \phi^k_{\eta_1}(z_2) & \cdots & \phi^k_{\eta_1}(z_N) \\ \phi^k_{\eta_2}(z_1) & \phi^k_{\eta_2}(z_2) & \cdots & \phi^k_{\eta_2}(z_N) \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \end{vmatrix}\prod_{i<j}^{N}(z_i - z_j)^{2p} \tag{5.2.11}$$

where we assume $k-1$ CF LL's are completely filled.

Then we use Laplace's determinant expansion to express the $\Psi_{k,2e}$ as a sum where each term is a product of a 2×2 determinant and a $N-2\times N-2$ determinant and all the factors from the top two lines of $\Psi_{k,2e}$ are contained in the former:

$$\begin{aligned}\Psi_{k,2e} = &\begin{vmatrix} \phi^k_{\eta_1}(z_1) & \phi^k_{\eta_2}(z_2) \\ \phi^k_{\eta_2}(z_1) & \phi^k_{\eta_1}(z_2) \end{vmatrix}|\cdots| - \begin{vmatrix} \phi^k_{\eta_1}(z_1) & \phi^k_{\eta_2}(z_3) \\ \phi^k_{\eta_3}(z_1) & \phi^k_{\eta_1}(z_3) \end{vmatrix}|\cdots| + \ldots = \\ &\begin{vmatrix} (\overline{\eta}_1 - \overline{z}_1)^k & (\overline{\eta}_1 - \overline{z}_2)^k \\ (\overline{\eta}_2 - \overline{z}_1)^k & (\overline{\eta}_2 - \overline{z}_2)^k \end{vmatrix}|\cdots| - \begin{vmatrix} (\overline{\eta}_1 - \overline{z}_1)^k & (\overline{\eta}_1 - \overline{z}_3)^k \\ (\overline{\eta}_2 - \overline{z}_1)^k & (\overline{\eta}_2 - \overline{z}_3)^k \end{vmatrix}|\cdots| + \ldots \end{aligned} \tag{5.2.12}$$

so that the determinants containing dots do not depend on η_1 or η_2.

To see what happens after the exchange we place the qe's opposite each other so that $\eta_1 = -\eta_2 \equiv \eta$. Then an interchange implies $\eta \to -\eta$ and the η-dependent determinants D_η transform as

$$\begin{aligned} D^{ij}_\eta \equiv (\overline{\eta} - \overline{z}_i)^k(-\overline{\eta} - \overline{z}_j)^k - (\overline{\eta} - \overline{z}_j)^k(-\overline{\eta} - \overline{z}_i)^k \longrightarrow \\ (-\overline{\eta} - \overline{z}_i)^k(\overline{\eta} - \overline{z}_j)^k - (-\overline{\eta} - \overline{z}_j)^k(\overline{\eta} - \overline{z}_i)^k = -D^{ij}_\eta \end{aligned} \tag{5.2.13}$$

This happens with every term so that $\Psi_{k,2e}$ as a whole changes sign. The flux attachment factor is unchanged while the projection is expected to leave topological properties intact. Thus, including the (sign-reversed) contribution from the Berry phase the statistical angle is

$$\theta_{qe} = \pi - \frac{2p\pi}{2pk+1} = \pi\frac{2p(k-1)+1}{2pk+1} \tag{5.2.14}$$

To find the qh angle we use the same trick as the one in section 4.2.3. When moving a CF qe around a qh the change in $\langle N\rangle_R$ has the opposite sign, because the expectation value is lower due to the qh. Thus the Berry phase attained after an exchange between a qe and a qh is $\Delta\gamma_B^{qe-qh} = -\Delta\gamma_B^{qe-qe} = \frac{2p2\pi}{2pk+1}$. The rest of the argument is the same as in section 4.2.3, giving that the Berry phases resulting from qe and qh interchange respectively are equal.

Since an absence of an electron, that is a hole, is also fermionic the qh's should have an additional contribution to the statistics too. For simplicity we consider a qh on the Laughlin sequence. According to [8] the wavefunction of a system that lacks two electrons to have a single filled LL, that is a system with two holes at η_1 and η_2, is $\Psi_{1,2h} = (\eta_1 - \eta_2)\prod_i(\eta_1 - z_i)(\eta_2 - z_i)\Psi_1$. Therefore the corresponding qh wavefunction is

$$\Psi^{cf}_{\nu,2qh} = \mathcal{P}_{LLL}(\eta_1 - \eta_2)\prod_{i=1}^{N}(\eta_1 - z_i)(\eta_2 - z_i)\Psi_1\prod_{j<k}^{N}(z_j - z_k)^{2p} \tag{5.2.15}$$

It is easy to see that the above has a monodromy that also adds a term π to the statistics angle. We assume that the same is true for a general qh so that we have

$$\theta_{qh} = \theta_{qe} = \pi \frac{2p(k-1)+1}{2pk+1} \tag{5.2.16}$$

The difference between (5.2.15) and Laughlin's proposal (4.2.6) is due to normalization. As we will see in section 6.2.1 the above is correctly normalized. But since we can change the phase of Ψ at will we may substitute $(\eta_1 - \eta_2) \to |\eta_1 - \eta_2|$. Then the complete statistics phase will show up as a Berry phase.

In this treatment we have ignored the possibility of unequal qh's. As long as the relevant qh has the same but opposite charge as the qe the combined qe-qh pair should be trivial and thus give no exchange phase (see section 7.3 for a discussion of the connection between charge and statistics). In these cases the above arguments are legitimate. Using other models we will find that at some filling factors qh's with different charges are indeed expected, but never on the Jain sequence, where the charge of the qe and all the qh's are equal in magnitude. Thus the value obtained above for the qh statistics angle is not affected by this caveat.

Numerical calculations of the CF qe statistical angle by Kjønsberg and Leinaas [39] and by Jeon and Jain [37] confirm the magnitude of (5.2.16) but with the opposite sign. Jeon and Jain explain this in [37] with the following reasoning:

To compute the Berry phase they consider a qe moving in a circle in the electron liquid and ask how it is altered when the loop contains another qe. But they show that the insertion of the latter particle does not leave the original qe trajectory unaffected - it pushes the first qe a bit outwards. When this is taken into consideration another term appears with the result that the change in Berry phase is exactly that of (5.2.16). With expressions for qp statistics arising from the CF scheme obtained we turn to field theory for one more model of the QHE.

Chapter 6

Conformal field theory

Conformal field theory is the latest of the approaches to describe the QHE that we will consider. In this chapter is an account with emphasis on the statistics angles.

6.1 Preliminaries

6.1.1 The connection between CFT and the QHE

From the late 1980's until the present ideas based on using conformal field theory (CFT) to study the QHE have been developed. In addition to creating trial wavefunctions for a wide variety of filling factors this scheme yields certain operators representing particles in the theory that often have crucial properties (like the charge and statistics of quasiparticles) explicitly coded. We shall not go in depth into this area but mainly use CFT as a cookbook recipe for wavefunctions; however a brief look at the relevant theory (based on [40, 41]) seems appropriate.

Although no conclusive, fundamental proof that correlators in a CFT should yield wavefunctions for QHE systems exists, there are compelling arguments to this effect, as originally suggested mainly by Moore and Read [42] and Wen [43]. An effective theory describing a two dimensional electron gas in a strong magnetic field akin to the bulk of the QHE system (effective in the sense that it describes the universal properties of the state) is found in the form of a Chern-Simons theory, which is a topological field theory (this aspect of the QHE is beyond our scope). In the case of the Laughlin states this connection is rigorously proven using mean field assumptions, and in more general cases strong arguments exist. The next logical step is the connection between the CS theory and CFT, and there are in fact two distinct (but related) paths to this.

In CS theory there are no massive degrees of freedom, so that when it is used to describe an electron gas the electron degrees of freedom are effectively integrated out. Consequently it does not offer a complete description of the QHE system, but quantities like the conductance and quasiparticle charge and statistics may be extracted. In fact anyonic quantities can be described in the theory: physically one may regard a quantity called Wilson loops as the world lines of anyons, also giving the latter's braiding phases. Wilson loops are defined as $W_C \equiv e^{\oint_C dx^\mu a_\mu}$ where a_μ is a gauge field, and Witten has shown [44] that expectation values of these equal correlators of anyonic versions of a kind of operators in CFT called vertex operators. It is therefore a natural guess that correlators of fermionic vertex operators in CFT

can be used to describe the electrons in the system (although as noted the electrons are not part of the CS theory and thus this guess is not proven).

When used in connection with the two-dimensional QHE the CS theory is defined in (2+1) dimensions (in Euclidian 3-space), but the Hamiltonian is equal to zero so that the dynamics are trivial (in conjunction with the fact there are no electron degrees of freedom). From this starting point one can do a formal transition into two dimensions of which both are spatial (since we are in Euclidian space there is no mathematical distinction), to describe equal-time slices of the QHE bulk. The dimensionality of the related CFT will be the same. This transition is very similar to the one used when utilizing a quantum field theory to describe statistical mechanics, where $it \to 1/kT$.

The second link connecting CS theory and CFT arises in the context of the edge of the QHE system. It can be shown that when we introduce edges the gauge invariance, and thereby the current conservation, of the CS theory describing the electron gas is lost. That is, when we add a divergence $\partial_\mu f$ to the gauge field a_μ the action is not invariant. To compensate one may demand $f = 0$ along the edge but this makes some degrees of freedom in the gauge field dynamical. As shown by Wen these are described by a chiral boson theory which can be implemented in a (1+1) dimensional CFT, called a Luttinger liquid (the edge is one-dimensional, so that this theory is dynamical with one temporal and one spatial dimension). This connection is rigorously proven.

A further conjecture has been made that the two CFT's describing the bulk and the edge of the QHE respectively are in fact the same theory. This is supported by all results up to date, thus the QHE constitutes a principle of holography.

Evidently there are several reasons to expect that one may use the CFT formalism to describe a QHE system. This in oppostion to the fact that the latter has no conformal invariance (there is a distance scale, namely the magnetic length l) and is not critical, as are the usual instances when CFT is appropriate[1]. As we will see the theory reproduces wavefunctions from both Laughlin's scheme and the CF model in addition to creating new ones (including non-abelian states which we will not consider).

However there is one caveat. The correlators are expected to give what is called representative wave functions, which are not always good ground state wave functions. In the simplest cases they are, but generally they are not, instead being adiabatically connected to such functions. This means that they can be changed continuously, keeping their topological properties (such as statistical phases) into ground state functions - and that they have the right topological properties to begin with.

As we will see the correlators we start with will sometimes be constructed from primary fields (to be defined later) and sometimes from fields called descendant fields, which can be constructed from the primary ones. An early suggestion by Moore and Read was that the space spanned by correlators of operators made from the primary field and descendant fields should contain the good wavefunctions, so that linear combinations of these can be used to create good wavefunctions. This is however problematic due to the fact that descendant fields contain derivatives which will change the angular momentum of the particles and thus the filling factor. Another proposal [32] is similar to what is sometimes done in CF, where one constructs linear combinations of ground state functions and functions containing a small number of excitons (bound qe-qh pairs and thus the smallest available excitations). These

[1] As mentioned one aspect of the QHE that is topological is the fact that the conductance measured is independent of the geometry of the sample.

are often seen to be closer to the Coloumb system in numerical calculations. The methods described above will however not be of relevance to us in that the original wavefunctions should have the correct statistical angles, which is our main concern.

6.1.2 The free boson

Before looking at the specifics of CFT we review some features of the free massless boson in a two dimensional quantum field theory, since our Lagrangian density will be $\mathcal{L} = \frac{1}{4\pi}\partial_\mu\phi(\mathbf{x})\partial^\mu\phi(\mathbf{x})$ where ϕ is bosonic and $\mathbf{x} \equiv (x^0, x^1)$. We place the field on a cylinder that is periodic in the x^1 direction: $\phi(x^0, x^1) = \phi(x^0, x^1 + L)$. Locally a cylinder is equivalent to flat space so that the metric tensor is unchanged. We have the action

$$S = \frac{1}{4\pi}\int d^2x\partial_\mu\phi\partial^\mu\phi = -\frac{1}{4\pi}\int d^2x\phi\partial^2\phi \tag{6.1.1}$$

where $\partial^2 \equiv \partial_\mu\partial^\mu$ and we have assumed that the fields disappear at the boundary. To calculate the propagator $K(\mathbf{x}, \mathbf{y}) \equiv \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle$ we use the fact that it is a Green's function for the differential operator in the action: $-\frac{1}{4\pi}\partial^2 K(\mathbf{x}, \mathbf{y}) = \delta^2(\mathbf{x} - \mathbf{y})$ [45] and that because of translational invariance we should have $K = K(r)$ where $r \equiv |\mathbf{x} - \mathbf{y}|$. We integrate over a disk Ω of radius r centered on y using polar coordinates:

$$\begin{aligned}\int d^2x\delta^2(\mathbf{x}-\mathbf{y}) = 1 = -\frac{1}{4\pi}\int_\Omega d^2x\partial^2 K(\mathbf{x}, \mathbf{y}) = -2\int_0^r dr'\ r'\frac{1}{r'}d_{r'}(r'd_{r'}K) = -2rd_rK \\ \implies \quad d_rK = -\frac{2}{r} \quad \implies \quad K(r) = -2\ln(r) = -2\ln|\mathbf{x} - \mathbf{y}|\end{aligned} \tag{6.1.2}$$

A central construction will be operators of the kind $V_i(x) : e^{i\gamma_i\phi_i(x)} :$ where $\gamma_i \in \mathbb{R}$, colons signify normal ordering and we now work in canonical quantization, so that ϕ are operator fields (we also look at one dimension only, the extension to two being trivial). These are charged objects that will represent particles. We Fourier decompose the fields and use annihilation and creation operators in the usual way:

$$\phi_i(x) = \sum_p \alpha_p^i a_p + \beta_p^i a_p^\dagger \quad \text{where} \quad [a_p, a_q^\dagger] = \delta_{pq} \tag{6.1.3}$$

Now we have $V_i(x) =: \prod_p e^{i\gamma_i(\alpha_p^i a_p + \beta_p^i a_p^\dagger)} :$. The normal ordered field should give zero vacuum expectation value, which using the relation $e^{\mathcal{O}} = \mathbb{1} + \mathcal{O} + \frac{1}{2}\mathcal{O}^2 + \ldots$ we see implies that V_i must have an expectation value of 1. To obtain this we write

$$: \prod_p e^{i\gamma_i(\alpha_p^i a_p + \beta_p^i a_p^\dagger)} := \prod_p e^{i\gamma_i\beta_p^i a_p^\dagger} e^{i\gamma_i\alpha_p^i a_p} \tag{6.1.4}$$

Next we need the Hausdorff relation $\exp(-A)B\exp(A) = B + [B, A] + 1/2[[B, A], A] + \ldots$ where A and B are operators. In our case the commutators are constants; using this we find

$$\begin{aligned}e^{-A}e^B e^A = e^{-A}(\mathbb{1} + B + \frac{1}{2}B^2\ldots)e^A = \mathbb{1} + B + [B, A] + \frac{1}{2}(B + [B, A])^2 + \ldots \\ = e^{B+[B,A]} = e^B e^{[B,A]} \quad \implies \quad e^A e^B = e^B e^A e^{[B,A]}\end{aligned} \tag{6.1.5}$$

where we have used $\exp(-A)B^2\exp(A) = \exp(-A)B\exp(A)\exp(-A)B\exp(A)$. Now we can treat a string of such operators, of the form $\cdots : \exp(\gamma_i(\alpha_p^i a_p + \beta_p^i a_p^\dagger)) :: \exp(\gamma_{i+1}(\alpha_p^{i+1} a_p + \beta_p^{i+1} a_p^\dagger)) : \cdots$. Using (6.1.3) to move the last factor containing $a_p^\dagger$ all the way to the left we get

$$: e^{i\gamma_1(\alpha_p^1 a_p + \beta_p^1 a_p^\dagger)} : \cdots : e^{i\gamma_n(\alpha_p^n a_p + \beta_p^n a_p^\dagger)} :=$$
$$e^{i\gamma_n\beta_p^n a_p^\dagger} : e^{i\gamma_1(\alpha_p^1 a_p + \beta_p^1 a_p^\dagger)} : \cdots : e^{i\gamma_{n-1}(\alpha_p^{n-1} a_p + \beta_p^{n-1} a_p^\dagger)} : e^{i\gamma_n\alpha_p^n a_p} e^{\beta_p^n(\alpha_p^1\gamma_1 + \ldots + \alpha_p^{n-1}\gamma_{n-1})} \quad (6.1.6)$$

Clearly, if we do this with all the creation operators we find

$$: e^{i\gamma_1(\alpha_p^1 a_p + \beta_p^1 a_p^\dagger)} : \cdots : e^{i\gamma_n(\alpha_p^n a_p + \beta_p^n a_p^\dagger)} := e^{i(\gamma_1\beta_p^1 + \ldots + \gamma_n\beta_p^n)a_p^\dagger} e^{i(\gamma_1\alpha_p^1 + \ldots + \gamma_n\alpha_p^n)a_p} e^{\sum_{i<j}^n \gamma_i\gamma_j\alpha_p^i\beta_p^j} \quad (6.1.7)$$

Finally, using (6.1.3), (6.1.2) and $\langle(\gamma_i(\alpha_p^i a_p + \beta_p^i a_p^\dagger))(\gamma_j(\alpha_p^j a_p + \beta_p^j a_p^\dagger))\rangle = \gamma_i\alpha_p^i\gamma_j\beta_p^j$ because of the orthonormality of states, and reinserting the product over p, we get

$$\langle V_1(x_1)\cdots V_n(x_n)\rangle = \langle\prod_p e^{i(\gamma_1\beta_p^1 + \ldots + \gamma_n\beta_p^n)a_p^\dagger} e^{i(\gamma_1\alpha_p^1 + \ldots + \gamma_n\alpha_p^n)a_p} e^{\sum_{i<j}^n \gamma_i\gamma_j\alpha_p^i\beta_p^j}\rangle =$$
$$\prod_p e^{-\sum_{i<j}^n \gamma_i\gamma_j\alpha_p^i\beta_p^j} = e^{-\sum_{i<j}^n \gamma_i\gamma_j\langle\phi_i(x_i)\phi_j(x_j)\rangle} = \prod_{i<j}^n (x_i - x_j)^{\gamma_i\gamma_j} \quad (6.1.8)$$

6.1.3 CFT

CFT is a quantum field theory that in addition to local gauge invariance includes the local continuous conformal symmetry, encompassing Lorentz symmetry, dilation symmetry and the special conformal symmetries (we do not need the specifics of these). In CFT it is common to work in imaginary time using a Wick rotation, that is to utlilize the time coordinate $x^0 \equiv -it$. This yields a Euclidian metric tensor $g_{\mu\nu} = diag\{1, \ldots, 1\}$ (in flat spacetime of course), blurring the distinction between space and time. Conformal symmetry means that the action integral is invariant under the following transformation:

$$g_{\mu\nu}(\mathbf{x}) \longrightarrow g'_{\mu\nu}(\mathbf{x}') = \Lambda(\mathbf{x})g_{\mu\nu}(\mathbf{x}) \quad (6.1.9)$$

where $\Lambda = 1$ gives the Poincare group. We note that the angle between vectors $\chi = -i\ln\left(\frac{g_{\mu\nu}x^\mu y^\nu}{|\mathbf{x}||\mathbf{y}|}\right) \pmod{2\pi}$ is preserved by (6.1.9). In two dimensions the conformal group is infinite dimensional and can be represented as continuous holomorphic mappings of complex coordinates z as demonstrated below:

A coordinate transformation $x^\mu \to y^\mu(\mathbf{x})$ changes the metric tensor in the following way:

$$g^{\mu\nu}(\mathbf{x}) \longrightarrow \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}g'^{\alpha\beta}(\mathbf{x}) = \Lambda(\mathbf{x})g^{\mu\nu}(\mathbf{x}) \quad (6.1.10)$$

Since the original and transformed tensors are proportional through (6.1.9) we have $g'^{\alpha\beta} = 0\ \forall\ \alpha \neq \beta$. Using this, (6.1.10) and the original euclidian tensor we get

$$\Lambda g^{00} = \Lambda = \left(\frac{\partial y^0}{\partial x^0}\right)^2 g'^{00} + \left(\frac{\partial y^0}{\partial x^1}\right)^2 g'^{11} = \Lambda g^{11} = \left(\frac{\partial y^1}{\partial x^0}\right)^2 g'^{00} + \left(\frac{\partial y^1}{\partial x^1}\right)^2 g'^{11}$$
$$\implies \quad \left(\frac{\partial y^0}{\partial x^0}\right)^2 + \left(\frac{\partial y^0}{\partial x^1}\right)^2 = \left(\frac{\partial y^1}{\partial x^0}\right)^2 + \left(\frac{\partial y^1}{\partial x^1}\right)^2 \quad (6.1.11)$$

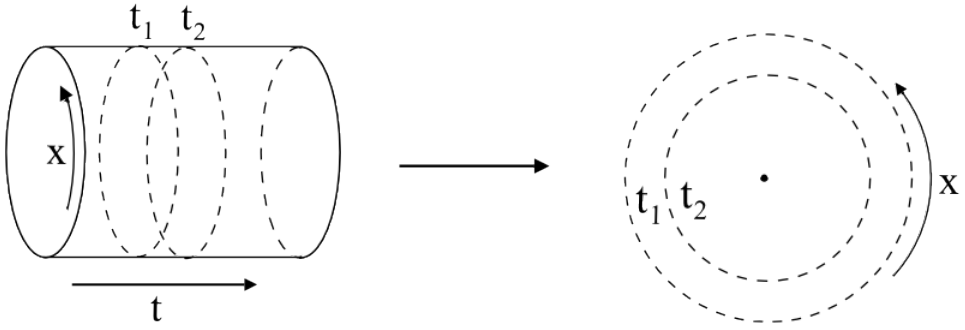

Figure 6.1: Mapping from the cylinder to the plane

and

$$\Lambda g^{01} = 0 = \frac{\partial y^0}{\partial x^0}\frac{\partial y^1}{\partial x^0} g'^{00} + \frac{\partial y^0}{\partial x^1}\frac{\partial y^1}{\partial x^1} g'^{11} \quad \Longrightarrow \quad \frac{\partial y^0}{\partial x^0}\frac{\partial y^1}{\partial x^0} + \frac{\partial y^0}{\partial x^1}\frac{\partial y^1}{\partial x^1} = 0 \tag{6.1.12}$$

(to be rigorous one needs to use the same argument with g^{10} to deduce the last equation). As can be shown (6.1.11) and (6.1.12) are equivalent to the Cauchy-Riemann equations, so that the conformal symmetry transformations in two dimensions can be represented by holomorphic functions - exactly the ones we use in generating the LLL wave functions in the QHE system. The group is infinite dimensional because a holomorphic function (and thus a group element) has infinitely many parameters as can be seen through the Laurent series.

The operator formalism, which is not manifestly Lorentz invariant and implies a specific frame of reference, will be utilized. It is beneficial to use what is called radial quantization, where time increases with distance from the origin. As an intermediate step to this goal we define the complex coordinate $\zeta \equiv \xi^0 + i\xi^1$. Now the Euclidian time coordinate ξ^0 increases along the axis while the spatial ξ^1 denotes position in the circumferential direction (these correspond to the previous x^0 and x^1). Then we map this cylinder onto the complex plane with coordinates $(z, \overline{z})$ using $z \equiv \exp(\frac{2\pi}{L}\xi)$ (see figure 6.1). Simultaneous points lie on circles of equal $|z|$ (since changing only ξ^1 in ζ gives an imaginary factor and thus only a phase change in z). Therefore, if two operators $A(z)$ and $B(z)$ are time-ordered, here equivalent to radial ordered, we have

$$\mathcal{R}(A(z)B(w)) = \begin{cases} A(z)B(w) & : \quad |z| > |w| \\ B(w)A(z) & : \quad |z| < |w| \end{cases} \tag{6.1.13}$$

where $\mathcal{R}$ signifies radial ordering. From now on correlators are defined as radial ordered instead of the usual time ordering.

We will use the complex coordinate $z = x^0 + ix^1$ and its complex conjugate $\overline{z}$ and when convenient write z^α where $z^0 \equiv z$ and $z^1 \equiv \overline{z}$. Note that x^0 and x^1 signify different quantities than in the discussions prior to the previous paragraph. z and $\overline{z}$ are treated as the independent coordinates instead of varying x^0 and x^1, but for the physical interpretations we require no dependence on $\overline{z}$ (we will mostly repress this dependency, and it must disappear when we consider quantities that should be holomorphic). x^0 is temporal within CFT but spatial when considering the QHE (the relationship between the two being complicated as indicated in the first section). We can find the metric tensor and its inverse expressed in z^α by using

conservation of the line element:

$$\begin{aligned} g_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu = (dx^0)^2 + (dx^1)^2 = g_{\alpha\beta}(z)dz^\alpha dz^\beta = g_{00}(z)((dx^0)^2 + 2idx^0dx^1 - (dx^1)^2) \\ + 2g_{01}(z)((dx^0)^2 + (dx^1)^2) + g_{11}(z)((dx^0)^2 - 2idx^0dx^1 - (dx^1)^2) \\ \implies \quad g_{\alpha\beta}(z) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \quad \implies \quad g^{\alpha\beta}(z) = g_{\alpha\beta}(z)^{-1} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \end{aligned} \tag{6.1.14}$$

A type of fields that will play a special role are primary fields. These are defined as those which transform under a local conformal transformation $z \to w(z)$ as

$$\Phi(z) \longrightarrow \left(\frac{\partial z}{\partial w}\right)^h \left(\frac{\partial \overline{z}}{\partial \overline{w}}\right)^{\overline{h}} \Phi(w) \tag{6.1.15}$$

and $(h, \overline{h})$ is called the conformal weight. The descendant fields discussed earlier are derivatives of primary fields. We need a special kind of primary field to make the connection with the QHE - the aforementioned vertex operators, defined as $V_i(z, \overline{z}) \equiv : e^{i\gamma_i\phi(z,\overline{z})} :$ (which is why we considered this kind of operator in the section concerning the free boson). It can be shown that $V(z, \overline{z}) = V(z) \otimes \overline{V}(\overline{z})$ where V and $\overline{V}$ are purely holomorphic and antiholomorphic respectively.

Using (6.1.8) we find

$$\begin{aligned} \langle \prod_{i=1}^N V_i(z_i, \overline{z}_i) \rangle = \langle : e^{i\gamma_1\phi(\mathbf{x}_1)} : \ldots : e^{i\gamma_N\phi(\mathbf{x}_N)} : \rangle = e^{\sum_{i<j}^N \langle -\gamma_i\phi(\mathbf{x}_i)\gamma_j\phi(\mathbf{x}_j)\rangle} = \\ \prod_{i<j}^N e^{2\gamma_i\gamma_j \ln|\mathbf{x}_i - \mathbf{x}_j|} = \prod_{i<j}^N |\mathbf{x}_i - \mathbf{x}_j|^{2\gamma_i\gamma_j} = \prod_{i<j}^N (z_i - z_j)^{\gamma_i\gamma_j} (\overline{z}_i - \overline{z}_j)^{\gamma_i\gamma_j} \end{aligned} \tag{6.1.16}$$

The result above is in keeping with the fact that V is a direct product of a holomorphic and an antiholomorphic operator. Since we want wavefunctions residing in the lowest LL we will use the holomorphic part only, and get

$$\langle \prod_{i=1}^N V_i(z_i) \rangle = \prod_{i<j}^N (z_i - z_j)^{\gamma_i\gamma_j} \tag{6.1.17}$$

From here on we will assume that all vertex operators are normal ordered without writing out the colons explicitly. We immediately see from (6.1.17) that we may reproduce the polynomial part of Laughlin's ground state wavefunctions for $\nu = 1/q$ with N electrons using $\gamma_i = \sqrt{q} \ \forall \ i$:

$$\langle e^{i\sqrt{q}\phi(z_1)} \ldots e^{i\sqrt{q}\phi(z_N)} \rangle = \prod_{i<j}^N (z_i - z_j)^q \tag{6.1.18}$$

Since q is an odd integer (making the operators fermionic) and we get one electron for each exponential we associate these vertex operators with electrons, each of the former creating one of the latter.

6.1.4 Electric charge and current

We note that the translational transformation $\phi \to \phi + a$ where a is a constant is a symmetry of the theory, since it leaves the action (6.1.1) invariant. In fact this is the U(1) symmetry of the vertex operators:

$$V(\mathbf{x}) = e^{i\gamma\phi(\mathbf{x})} \longrightarrow e^{i\chi}V(\mathbf{x}) \quad \Longrightarrow \quad \phi(\mathbf{x}) \longrightarrow \phi(\mathbf{x}) + \frac{\chi}{\gamma} = \phi(\mathbf{x}) + a \tag{6.1.19}$$

From Noether's theorem [45] the corresponding conserved current (whose significance we will ascertain later) is

$$j_a^\mu \equiv \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta^\mu_\nu\right)\frac{\delta x^\nu}{\delta\omega_a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\delta\Phi'(\mathbf{x}')}{\delta\omega_a} = -\frac{1}{2\pi}\partial^\mu\phi \tag{6.1.20}$$

We find the holomorphic part of this using (6.1.14):

$$J(z) \equiv 2\pi j_0(z) = 2\pi g_{0\alpha}(z)\frac{\partial z^\alpha}{\partial x^\mu}j^\mu(\mathbf{x}) =$$
$$-2\pi\frac{1}{2\pi}\left(\frac{1}{2}\frac{\partial\overline{z}}{\partial x^0}i(\partial_{\overline{z}} - \partial_z)\phi(z) + \frac{1}{2}\frac{\partial\overline{z}}{\partial x^1}(\partial_z + \partial_{\overline{z}})\phi(z)\right) = i\partial_z\phi(z) \tag{6.1.21}$$

We then define a charge operator $Q \equiv \frac{1}{2\pi i}\oint dzJ(z)$ where the contour is a circle around the origin and thus integrates over all space at a given time cf. the radial ordering. It can be shown that Q commutes with the hamiltonian so that the charge is conserved. We calculate Q_A, the charge eigenvalue of a state $|A\rangle \equiv A|0\rangle$ where $|0\rangle$ is the vacuum, using the following commutator:

$$[Q, A]|0\rangle = QA|0\rangle - AQ|0\rangle = Q|A\rangle = Q_A|A\rangle \quad \Longrightarrow \quad Q_A A = [Q, A] \tag{6.1.22}$$

where we have used that the vacuum is neutral.

Next we consider the integral $\oint_w dza(z)b(w)$ where a and b are radially ordered operators (so that they may be used as they stand inside a correlator) and the integration path is a counterclockwise circle around w. This integral can be expressed as the difference between two others that are circles centered on the origin (see figure 6.2. The paths do not have to be precisely equal if the integrands are holomorphic outside w, through Cauchy's theorem):

$$\oint_w dza(z)b(w) = \oint_{C_1} dza(z)b(w) - \oint_{C_2} dza(z)b(w) = [A, b(w)] \tag{6.1.23}$$

where, since b is independent of z, we have defined $A \equiv \oint dza(z)$. The integration is along a circle about the origin and therefore at constant time.

A central concept in CFT but one we will not make much use of is the operator product expansion. This is a representation of the product of two operators at different spacetime points expressed as a sum of terms including a single operator each and valid as the two points approach each other. We will need the operator product expansion of $\partial_z\phi$ and V, which is [40]

$$\partial_z\phi(z)V(w) \sim \frac{\gamma}{i}\frac{V(w)}{z - w} \tag{6.1.24}$$

where γ is the scalar in the exponent of V and $\sim$ signifies that the expression is modulo terms that are regular as $z \to w$.

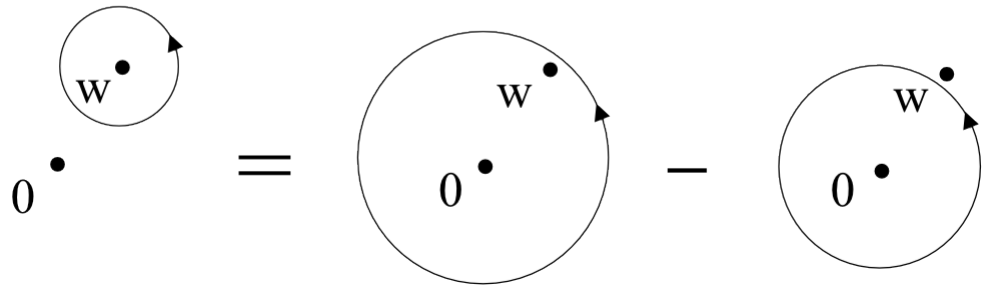

Figure 6.2: Subtraction of contours to be used with Cauchy's integral theorem

Now we may use (6.1.23) and (6.1.24) to calculate the charge of the electron operators:

$$Q_V V(z') = [Q, V(z')] = \frac{1}{2\pi i}\oint_{z'} dz J(z)V(z') =$$
$$\frac{1}{2\pi}\oint_{z'} dz\partial_z\phi(z)V(z') = \frac{\sqrt{\gamma}}{2\pi i}\oint_{z'} dz\frac{V(z')}{z-z'} = \sqrt{\gamma}V(z') \tag{6.1.25}$$

where we have used the residue theorem so that none of the regular terms not included in the operator product expansion would have given any contribution. Thus we see that the charge of the electron operators is $\sqrt{\gamma}$. Looking at the derivation above we note that the charge of a product of vertex operators is the sum of the individual charges. We want to express charge in terms of the charge of the electron vertex operators, and therefore we modify the current operator:

$$J(z) \equiv \frac{i}{\sqrt{\gamma}}\partial_z\phi(z) \tag{6.1.26}$$

so that V has unit U(1) charge $Q_V = 1$. Now it is time to find the physical meaning of this charge in the context of the QHE. We obtained it through the U(1) symmetry of vertex operators, and it corresponds to vorticity [46], so that a positive vorticity corresponds to a local depletion of the electron liquid and vice versa. Therefore, when we measure local electric charge Q_E relative to the homogeneous ground state charge of the liquid we should find

$$Q_E = -e(\Delta N - Q) \tag{6.1.27}$$

where ΔN is number of electrons added by the operator relative to the ground state. This means that Q_E of V is zero, which makes sense since these operators are used in fabricating the ground state and should create no local charge variations.

6.1.5 The neutrality condition

However there is one point we have glossed over at the end of section 6.1.3 when claiming that the vertex operators are direct products of a holomorphic and an antiholomorphic part. As usual the boson fields may be decomposed into Fourier modes; the resulting expression is [40]

$$\phi(z,\overline{z}) = \phi_0 - i\pi_0\ln(z\overline{z}) + i\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + a_n^\dagger \overline{z}^{-n}\right) \tag{6.1.28}$$

where a_n and $a_n^\dagger$ are the Fourier coefficients (and after quantization the creation and annihilation operators) and $\pi_n \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n}$ are the canonical momentum modes. We immediately see that the zero mode ϕ_0 has a special role. This complicates the division into a holomorphic and an antiholomorphic part of the vertex operators V. It can be shown [41] that we must also include a normal ordering: $V(z,\overline{z}) =: V(z) \otimes \overline{V}(\overline{z}) :$. This has important consequences, among them a constraint on vertex operators inside correlators, which is demonstrated (without reference to the zero mode) below:

The correlators also have to be translationally invariant, that is to be unchanged in the process $\phi \to \phi + a$. But we see that this is not generally the case for the holomorphic $V_i(z) = e^{i\gamma_i \phi(z)}$:

$$\langle \prod_i V_i(z_i) \rangle \longrightarrow \langle \prod_i V(z_i) \rangle e^{ia \sum_i \gamma_i} \tag{6.1.29}$$

Therefore, to have translational invariance we must demand $\sum_i \gamma_i = 0$. Since the γ's give the charge of the operators this is known as the neutrality condition.

But this means that (6.1.18) vanishes, and it turns out that we have to include a neutralizing background charge to avoid this. In the context of the QHE this is identified with the ion lattice, and it is encouraging that the CFT formalism demands its presence. There are several ways to include the background (which we will not go into here), and the one used in [47] gives the exponential factors $\exp(\frac{-1}{4l^2}\sum_i^N |z_i|^2)$ so that we get the complete wavefunctions for $\nu = 1/q$. In the remaining we will not discuss the background charge and suppress the ever present exponential factors.

6.2 FQHE wavefunctions from CFT

6.2.1 Laughlin quasiholes

As noted in (6.1.18) we may produce the Laughlin wavefunctions from vertex operators $V(z) = e^{i\sqrt{q}\phi(z)}$:

$$\langle \prod_{i=1}^N V(z_i) \rangle = \prod_{i<j}^N (z_i - z_j)^q \tag{6.2.1}$$

It is not hard to see how to recreate the Laughlin qh wavefunctions in a similar manner. We see from (6.1.16) that to create m qh's we can insert m vertex operators $H(\eta) \equiv \exp \frac{i}{\sqrt{q}}\phi(\eta)$:

$$\begin{aligned} \langle H(\eta_1) \dots H(\eta_m) \prod_{j=1}^N V_1(z_j) \rangle &= \langle e^{\frac{i}{\sqrt{q}}\phi_1(\eta_1)} \dots e^{\frac{i}{\sqrt{q}}\phi_1(\eta_m)} \prod_{j=1}^N e^{i\sqrt{q}\phi_1(z_j)} \rangle \\ &= \prod_{j<l}^m (\eta_j - \eta_l)^{\frac{1}{q}} \prod_{s=1}^N \prod_{t=1}^m (z_s - \eta_t) \prod_{u<v}^N (z_u - z_v)^q \end{aligned} \tag{6.2.2}$$

Next we calculate the U(1) charge of H using the modified (6.1.20) and (6.1.24) with $\gamma \to 1/\gamma$:

$$[Q, H(\eta)] = \frac{1}{2\pi\sqrt{q}} \oint_\eta dz \partial_z \phi H(\eta) = \frac{1}{2\pi i q} \oint_\eta dz \frac{H(\eta)}{z - \eta} = \frac{1}{q} H(\eta) \quad \Longrightarrow \quad Q_H = \frac{1}{q} \tag{6.2.3}$$

Thus the local electric charge of the qh is $Q_E = -e(\Delta N - Q) = -e(0 - 1/q) = e/q$ in agreement with the other models.

If we define the relative $\eta_{ij} \equiv \eta_i - \eta_j \equiv \rho_{ij} e^{i\chi_{ij}}$ it is easy to see from (6.2.2) that a clockwise permutation of two qh's (which amounts to $\chi_{ij} \to \chi_{ij} + \pi$) gives the expexted phase $\exp(i\frac{\pi}{q})$. Thus the operators have a monodromy. But we cannot be certain that an additional statistical phase will not surface as a Berry phase. To normalize the wavefunction we use the plasma analogy. Reinserting the exponential factors we have

$$\Psi_L^{2qh}\Big|_{norm} = \mathcal{N}(\eta_1 - \eta_2)^{\frac{1}{q}} \prod_{i=1}^{N}(z_i - \eta_1)(z_i - \eta_2) \prod_{i<j}^{N}(z_i - z_j)^q e^{-\frac{1}{4l^2}\sum_i^n |z_i|^2} \tag{6.2.4}$$

where $\mathcal{N}$ is the normalization constant we are after. As before we use the absolute square of this as the partition function for a plasma with potential energy U and inverse temperature $\beta = 2/q$. We then get

$$\begin{aligned} U = &- \ln|\eta_1 - \eta_2| - q\sum_{i=1}^{n}(\ln|z_i - \eta_1| + \ln|z_i - \eta_2|) \\ &- q^2 \sum_{i<j}^{n} \ln|z_i - z_j| + \frac{q}{4l^2}\sum_{i=1}^{n}|z_i| - \frac{q}{2}\ln|\mathcal{N}|^2 \end{aligned} \tag{6.2.5}$$

Again we have particles of charge q interacting with each other and with a uniform background of charge density $\rho = -1/2\pi l^2$ through a two-dimensional Coloumb potential. In addition there are now two particles of unit charge interacting with each other and the electrons, and a term containing $\mathcal{N}$. It can be shown [25] that the spatial probability distribution of the plasma particles is independent of η_1 and η_2 [2] through screening by the electron liquid if the charges described by η_1 and η_2 also interact with the background charge. To achieve this we use $\mathcal{N} \equiv \tilde{N}\exp\left(-\frac{1}{4ql^2}(|\eta_1|^2 + |\eta_2|^2)\right)$ where $\tilde{N}$ is independent of $\{z_i\}$, η_1 and η_2.

For ease of calculation we reduce to the case $\eta_1 = -\eta_2 \equiv \eta = Re^{i\theta}$ so that, writing Ψ for $\Psi^L_{1/q,2qe}\Big|_{norm}$:

$$\Psi = \tilde{N} 2^{\frac{1}{q}} e^{-\frac{1}{2ql^2}|\eta|^2} \eta^{\frac{1}{q}} \prod_{j=1}^{N}(z_j - \eta)(z_j + \eta)\Psi^l_{1/q} \tag{6.2.6}$$

where $\Psi^l_{1/q}$ is independent of η. We consider a complete anticlockwise rotation; as in section 4.2.2 the resulting Berry phase is given by

$$\gamma_B = i\int_{t(\theta=0)}^{t(\theta=2\pi)} dt \langle\Psi(\theta)|\partial_\theta\Psi(\theta)\rangle \frac{d\theta}{dt} = \int_0^{2\pi}\langle\Psi|i\partial_\theta\Psi\rangle d\theta \tag{6.2.7}$$

Next we utilize $i\partial_\theta = i\frac{\partial\eta}{\partial\theta}\partial_\eta + i\frac{\partial\overline{\eta}}{\partial\theta}\partial_{\overline{\eta}} = \overline{\eta}\partial_{\overline{\eta}} - \eta\partial_\eta$ and find

$$\begin{aligned} &\partial_\eta\Psi = \frac{1}{q\eta}\Psi - \frac{1}{2ql^2}\overline{\eta}\Psi + \sum_i\left(\frac{\Psi}{z_i + \eta} - \frac{\Psi}{z_i - \eta}\right) \qquad \partial_{\overline{\eta}}\Psi = -\frac{1}{2ql^2}\eta\Psi \\ &\implies \quad i\partial_\theta\Psi = (\overline{\eta}\partial_{\overline{\eta}} - \eta\partial_\eta)\Psi = \frac{1}{q}\Psi + \eta\sum_i\left(\frac{\Psi}{z_i + \eta} - \frac{\Psi}{z_i - \eta}\right) \end{aligned} \tag{6.2.8}$$

[2] This is required because there is no a priori preferred location for the qh's.

The first term originates from the monodromy factor. It gives rise to the following Berry phase:

$$\gamma_B^{mon} = \int_0^{2\pi} \langle \Psi | \frac{1}{q} \Psi \rangle d\theta = \frac{2\pi}{q} \tag{6.2.9}$$

where we used the fact that Ψ is normalized. This term is of equal magnitude but opposite sign from the statistics phase we extracted following Arovas et al in section 4.2.2. This means that these terms will cancel and the only Berry phase is the usual Aharonov-Bohm phase resulting from charged particles moving in a magnetic field - the monodromy gives the total statistics angle.

Nayak and Wilczek [48] and others have argued that wave functions given as CFT correlators of electron and qh vertex operators generally have no statistical contribution to the Berry phase *if* all fields in the qh operators are contained in at least one of the electron operators. The last statement is relevant because we may introduce additional non-charged bosonic fields in the qh operators only, which remove the fractional exponents. The missing monodromy phase then appears as a Berry phase so that the statistical phase (the sum of both) is unchanged [47]. When we do not have such additional fields we thus assume that the complete statistical phases may be read out from the monodromy.

Using the same logic as above one would assume that to create qe's we simply use a vertex operator $H^{-1}(\eta) = \exp\left(-\frac{i}{\sqrt{q}}\phi(\eta)\right)$. But this leads to a singularity when $\eta = z$ and z is an electron coordinate, in keeping with the Pauli principle - there is a finite probability that there already is an electron at η so that we can not make the electron liquid denser here using an antivortex.

A solution is, instead of inserting a new qe operator, to modify the existing electron operators. What is done is to shrink the correlation hole around all electrons and then create a coherent superposition around the wanted qe position, making the electron liquid denser close to this. Among other things we must lower the exponent, which indeed is accomplished by multiplying H^{-1} onto each electron (at the same position), creating excess charge around them. This procedure is described in [47] and results in a quasilocal operator $\mathcal{P}(\overline{\eta})$ which creates a qe localized within $\sim l$ of $\overline{\eta}$.

6.2.2 The Jain sequence

In a later section we will discuss Haldane and Halperin's hierarchy construction in the context of CFT. This entails creating condensates with the aforementioned $\mathcal{P}$, since we need local qp's (we only look at qe condensates and not those of qh's, for reasons explained later). In this section however we will use a different procedure described in [46] although this has a slightly more ad hoc character, for the reason that it is simpler. This will result in wavefunctions for states on the positive Jain series, the negative requiring condensing of qh's.

To create a qe we modify one of the electron operators in a Laughlin state by multiplying with an inverse hole, the latter being an antivortex and therefore shrinking the correlation hole:

$$V(z) = e^{i\sqrt{q}\phi(z)} \longrightarrow P(z) \equiv V(z)H^{-1}(z) = \partial_z e^{i(\sqrt{q}-1/\sqrt{q})\phi(z)} \tag{6.2.10}$$

where we have put in a derivative by hand, because otherwise we would end up with a wavefunction equal to zero. The U(1) charge is $Q_P = (1 - 1/q)$ which makes the electric charge $Q_E = -e(\Delta N - Q) = -e(1 - 1 + 1/q) = -e/q$; the charge we expect from other models.

The wavefunction has to be antisymmetric in electron coordinates, which a simple insertion of P will not accomplish. Therefore we use an antisymmetrization operator $\mathcal{A}$, making the wavefunction nonlocal in the qe position. There is no singularity in the resulting wavefunction (ignoring all exponential factors as usual):

$$\Psi^{cft}_{1/q,qe} = \mathcal{A}\left(\langle P(z_1)\prod_{i=2}^{N} V(z_i)\rangle\right) = \langle \begin{vmatrix} P(z_1) & V(z_1) & \dots & V(z_1) \\ P(z_2) & V(z_2) & \dots & V(z_2) \\ \vdots & & \ddots & \vdots \\ P(z_N) & V(z_N) & \dots & V(z_N) \end{vmatrix} \rangle =$$

$$\sum_{i=1}^{N}(-1)^{i+1}\langle P(z_i)\prod_{k\neq i}^{N} V(z_k)\rangle = \sum_{i=1}^{N}\partial_i(-1)^{i+1}\prod_{k<l}^{\neq i}(z_k-z_l)^q\prod_{k}^{\neq i}(z_k-z_i)^{q-1} \tag{6.2.11}$$

where we have used that correlators are associative and that the differential operator can be taken outside of the correlator, and expanded by the first row in the determinant. As shown in [46] this is equal to the CF wavefunction for a qe with sharp angular momentum zero; this momentum can also be calculated numerically. To give the qe angular momentum l one may insert a factor z^{l+1}. The exponent can be understood in the context of CF's by noting that the qe is residing in the second CF LL. The main difference with this approach and the one using $\mathcal{P}$ is that the latter is (quasi)local, while to get a local qe using P one must make a superposition of different angular momenta.

Continuing to let the CF theory guide us, in analogy we expect the wavefunction for m qe's to be of the form $\mathcal{A}\left(f_m\langle P(z_1)\cdots P(z_m)\prod_{i=m+1}^{m+N}V(z_i)\rangle\right)$ where f_m takes care of the relative angular momentum (we neglect the insignificant center of mass angular momentum). We will then again get equal wavefunctions to the ones arising with CF's provided that we take the following two measures:

Firstly we must move the differentials all the way to the left. Since they were introduced in an ad hoc manner anyway this is not a significant alteration. Secondly the angular momentum factor should be chosen as

$$f_m \equiv \prod_{a<b}^{m}(z_a-z_b)^{1-\frac{1}{q}} \tag{6.2.12}$$

the term $-1/q$ in the exponent can be explained as follows: Contraction of qe operators will give factors $\sim (z_a-z_b)^{q-2+1/q}$. But the wavefunction must be holomorphic in electron coordinates, hence the canceling term in f_m. Analogous to the treatment of a single qe one may show [46] that the relative angular momenta of these qe's are the lowest possible, namely $j_{ab}=1$. Thus if we look at the qe's in the wavefunction as a hierarchical condensate as in section 4.3 the density is the highest possible (since lower relative angular momenta brings particles closer together). This turns out to be true on the whole of the Jain sequence; it can be seen as a maximum density qp condensate.

By inserting f_m to gain holomorphic functions antisymmetric in the z's we remove the explicit monodromy contained in exponential factors $1/q$. Therefore the expected statistics must have been moved to the Berry phase if this method is sound. This phase has not been calculated analytically but it is confirmed by numerical calculations. Similarly, when using the quasi-local qe operator $\mathcal{P}$ one must introduce non-charged fields in the H^{-1} used to create $\mathcal{P}$. This will make the operator bosonic or fermionic, to obtain holomorphic functions[3]. Thus

[3]This is analogous to the mentioned trensfer of θ from the monodromy to the Berry phase of H.

the incorporation of H^{-1} in the qe operator is suggestive of fractional statistics but the issue is not straightforward.

Using a theorem generalizing the expansion of a row in (6.2.11) when dealing with antisymmetrization [46] we then have the (unnormalized) wavefunction for $N+m$ electrons and m qe's all in the lowest angular momentum $j=1$:

$$\Psi^{cft}_{1/q,mqe} = \mathcal{A}\left(\prod_{a<b}^{m}(z_a-z_b)^{1+j_{ab}-1/q}\langle\prod_{a=1}^{m}P(z_a)\prod_{i=m}^{m+N}V(z_i)\rangle\right) = \sum_{i_1<\ldots<i_m}(-1)^{\sum_l^m i_l}\prod_{a<b}^{m}(z_{i_a}-z_{i_b})^{2-1/q}\partial_{i_1}\cdots\partial_{i_m}\prod_{a<b}^{m}(z_{i_a}-z_{i_b})^{q-2+1/q} \prod_{c_1}^{\neq i_2,\ldots,i_m}(z_{c_1}-z_{i_1})^{q-1}\cdots\prod_{c_m}^{\neq i_1,\ldots,i_{m-1}}(z_{c_m}-z_{i_m})^{q-1}\prod_{i<l}^{\neq i_1,\ldots i_m}(z_i-z_l)^q \tag{6.2.13}$$

The first parenthesis comes from f_m, the second from contracting the qe operators among themselves, the ensuing product of products comes from contracting each qe operator with each electron operator, and the last from contracting electron operators among themselves.

We have now seen how we can find CF wavefunctions starting from CFT and with an approach inspired by Haldane and Halperin's hierarchy. But they can also be obtained by introducing a second independent bosonic free field ϕ_2 (labeling the original field and operators ϕ_1, V_1 and P_1 respectively) and defining a new electron vertex operator

$$V_2(z) \equiv \partial_z e^{\frac{(q-1)i}{\sqrt{q}}\phi_1(z)}e^{\frac{(2q-1)i}{\sqrt{q(2q-1)}}\phi_2(z)} \tag{6.2.14}$$

The correlator of an equal number of the two operators gives the desired expression:

$$\mathcal{A}\langle\prod_{1_2=1}^{N/2}V_2(z_{i_2})\prod_{i_1=1}^{N/2}V_1(z_{i_1})\rangle = \mathcal{A}\left(\langle\prod_{i_2=1}^{N/2}\partial_{i_2}e^{\frac{(q-1)i}{\sqrt{q}}\phi_1(z_{i_2})}e^{\frac{(2q-1)i}{\sqrt{q(2q-1)}}\phi_2(z_{i_2})}\prod_{i_1=1}^{N/2}e^{i\sqrt{q}\phi_1(z_{i_1})}\rangle\right) = \mathcal{A}\left(\prod_{i_2=1}^{N/2}\partial_{i_2}\prod_{i_2<j_2}^{N/2}(z_{i_2}-z_{j_2})^q\prod_{i_2=1}^{N/2}\prod_{i_2=1}^{N/2}(z_{i_2}-z_{i_1})^{q-1}\prod_{i_1<j_1}^{N/2}(z_{i_1}-z_{j_1})^q\right) \tag{6.2.15}$$

where we have used that the different fields commute. Without explicitly writing out the antisymmetrized sum one can recognize the result as equivalent to the CF result [46]. Intuitively the construction is also similar to the CF model, with an equal number of two different kinds of electrons (corresponding to composite fermions in two different CF Landau levels) being antisymmetrized over. With this we have started with a hierarchy picture and obtained a CF type expression, hinting at a (controversial) connection between the two. It also indicates that the CF LL k and the hierarchy level n are associated. This connection is even clearer when using the approach described in section 6.2.3.

Since we will encounter several wavefunctions similar to (6.2.15) we introduce a compact

notation following [46]. we define the following factors:

$$\partial_\alpha^{\alpha-1} \equiv \prod_{i_\alpha=1}^{N_\alpha} \partial_z^{\alpha-1} \qquad (\alpha-\alpha) \equiv \prod_{i_\alpha<j_\alpha}^{N_\alpha} (z_{i_\alpha} - z_{j_\alpha})$$
$$(\alpha-\beta) \equiv \prod_{i_\alpha=1}^{N_\alpha}\prod_{i_\beta=1}^{N_\beta} (z_{i_\alpha} - z_{i_\beta}) \quad : \ \alpha \neq \beta \tag{6.2.16}$$

where N_α denote the number of electrons in the corresponding group or hierarchy level. The N_α's are thus not explicitly shown when the wavefunctions are written in this form. (6.2.15) can then be written as follows:

$$\Psi_{\nu_2}^{cft} = \mathcal{A}\left(\partial_2(2-2)^q(2-1)^{q-1}(1-1)^q\right) \tag{6.2.17}$$

where ν_2 signifies that we are on the second hierarchy level.

As things are standing the electric charge of V_2 is not unity however. The system density is altered and we must change the background charge density accordingly. To this end we once again redefine the U(1) charge operator:

$$J(z) \equiv \frac{i}{\sqrt{q}}\partial_z\phi_1(z) + \frac{i}{\sqrt{q(2q-1)}}\partial_z\phi_2(z) \tag{6.2.18}$$

This gives V_2 a charge of unity. As shown in [46] the filling factor can be obtained from the current operator. This is done using the plasma analogy and the exponential factors (neglected in our case), and considering the background electron density. The result is

$$\text{if} \quad J(z) = \sum_{\alpha=1}^{n} \frac{\partial_z\phi_\alpha}{R_\alpha} \quad \text{then} \quad \nu = \sum_{\alpha=1}^{n} \frac{1}{R_\alpha^2} \tag{6.2.19}$$

At level two of our maximum density system we then have $\nu = \frac{1}{q} + \frac{1}{q(2q-1)} = \frac{2}{2q-1}$.

As in the CF scheme there are two different qh operators in this level, making a (partial) vortex with each of the two kinds of electrons. It can be shown [49] that we must have

$$\langle H_i V_j \rangle \sim (z_i - z_j)^{\delta_{ij}} \tag{6.2.20}$$

This leads to the following qh operators (where the superscript indicates hierarchy level):

$$H_1^2(z) \equiv e^{\frac{i}{\sqrt{q}}\phi_1(z)} e^{\frac{(1-q)i}{\sqrt{q(2q-1)}}\phi_2(z)} \qquad H_2^2(z) \equiv e^{\frac{qi}{\sqrt{q(2q-1)}}\phi_2(z)} \tag{6.2.21}$$

These operators have the following charges:

$$[Q(z), H_1^2(w)] = \left[\frac{1}{\sqrt{q}}\oint dz\partial_z\phi_1(z)e^{\frac{i}{\sqrt{q}}\phi_1(w)}\right] e^{\frac{(1-q)i}{\sqrt{q(2q-1)}}\phi_2(w)} +$$
$$\left[\frac{1}{\sqrt{q(2q-1)}}\oint dz\partial_z\phi_2(z)e^{\frac{(1-q)i}{\sqrt{q(2q-1)}}\phi_2(w)}\right] e^{\frac{i}{\sqrt{q}}\phi_1(w)} =$$
$$\left(\frac{1}{q} + \frac{1-q}{q(2q-1)}\right) H_1^2(w) = \frac{1}{2q-1}H_1^2(w) \quad \Longrightarrow \quad Q_{H_1^2} = \frac{1}{2q-1}$$
$$[Q(z), H_2^2(w)] = \ldots = \frac{1}{2q-1}H_2^2(w) \quad \Longrightarrow \quad Q_{H_2^2} = \frac{1}{2q-1} \tag{6.2.22}$$

In finding the monodromy and thus the statistics we create wavefunctions for systems of two qh's - two of type 1, two of type 2 and lastly one of each[4]. However it suffices to calculate the correlator of the qh operators only, since all of the nontrivial phase comes from from these:

$$\begin{aligned}
\langle H_1^2(\eta)H_1^2(\eta')\rangle = (\eta-\eta')^{\frac{1}{q}}(\eta-\eta')^{\frac{(1-q)^2}{q(2q-1)}} \quad &\Longrightarrow \quad \theta_{H_1} = \frac{\pi q}{2q-1} \\
\langle H_2^2(\eta)H_2^2(\eta')\rangle = (\eta-\eta')^{\frac{q}{2q-1}} \quad &\Longrightarrow \quad \theta_{H_2} = \frac{\pi q}{2q-1} \\
\langle H_1^2(\eta)H_2^2(\eta')\rangle = (\eta-\eta')^{\frac{1-q}{2q-1}} \quad &\Longrightarrow \quad \theta_{H_1\leftrightarrow H_2} = \frac{\pi(1-q)}{2q-1}
\end{aligned} \tag{6.2.23}$$

These qh's are in a way partial vortices, in that they multiply factors $(\eta - z_i)$ but on groups of electron coordinates z_i instead if all of them like Laughlin qh's. However if we place one H_1 and one H_2 at the same position they will together multiply such factors on all electrons. This indicates that the resulting composite is a Laughlin qh, and explicit calculations show that the combined vortex operator has electric charge $Q_E = \nu e$ and monodromy $\theta = \nu\pi$. This generalizes to any filling factor: a vortex operator consisting of one each of the qh operators, $H_1 H_2 \cdots H_n$ consitutes a Laughlin qh.

We can imagine creating various qh condensates in this fashion. If we thereby construct an operator with the same exponents as an electron vertex operators (but thus with the opposite electric charge cf. (6.1.27) using $\Delta N = 0$) we would intuitively think that these are the electrons' antiparticles. As shown by explicit calculations, various combinations of qh operators with total charge $Q = e$ (for example $H_1^2 H_2^3$ at $\nu = 2/5$) are fermions.

To advance to the next level in the hierarchy we find the qe operator in the second level. With the same reasoning as before there are now four candidates for this operator, in combining different inverse qh and electron operators. But as can be shown [**??**] three of these give a wavefunction equal to zero, leaving us with the one combining V_2 and $(H_2^2)^{-1}$:

$$P^2(z) \equiv \partial_z^2 e^{\frac{(q-1)i}{\sqrt{q}}\phi_1(z)} e^{\frac{(q-1)i}{\sqrt{q(2q-1)}}\phi_2(z)} \tag{6.2.24}$$

where we have to insert another differential operator so that the correlators will not equal zero. This is also in agreement with the CF model, where there is only one qe operator per ground state (but in that case because the smallest excitation correspondong to a qe is the unique operation of adding one CF to the next CF LL). The implication is that concerning the electron and qh operators the level index n corresponds to the CF LL $k = n$ but that the CFT qe operators we have labelled n reside in the CF LL $n+1$. Using the same logic as earlier we can construct a maximum density qe condensate and move the differentials to the left, or introduce a new field $\phi_3(z)$ and electron operator $V_3(z) = P^2(z)\exp(iC\phi_3(z))$ that will create the same expression. Since the contraction of the latter with itself must give an exponent q the constant C is:

$$\frac{(q-1)^2}{q} + \frac{(q-1)^2}{q(2q-1)} + C^2 = q \quad \Longrightarrow \quad C = \frac{3q-2}{\sqrt{(3q-2)(2q-1)}} \tag{6.2.25}$$

Now we have the new electron operator, and the current operator must be modified accordingly with an additional term $\partial_z\phi_3(z)/\sqrt{(3q-2)(2q-1)}$. This leads to the filling factor

[4]This means that we are considering statistics between different particles. For comments on this see section 7.1

$\nu = \frac{3}{3q-2}$. Using (6.2.20) once again we can also find the qh operators, and with similar calculations as earlier we find the U(1) charges $Q_{H_1^3} = Q_{H_2^3} = Q_{H_3^3} = \frac{1}{3q-2}$ and monodromies $\theta_{H_1} = \theta_{H_2} = \theta_{H_3} = \frac{\pi(2q-1)}{3q-2}$ and $\theta_{H_1 \leftrightarrow H_2} = \theta_{H_1 \leftrightarrow H_3} = \theta_{H_2 \leftrightarrow H_3} = \frac{\pi(1-q)}{3q-2}$.

As an example of the above we consider a maximum density system at level three with $q = 3$. This gives us

$$J(z) = \frac{\partial_z \phi_1}{\sqrt{3}} + \frac{\partial_z \phi_2}{\sqrt{15}} + \frac{\partial_z \phi_3}{\sqrt{35}} \quad \Longrightarrow \quad \nu = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} = \frac{3}{7} \tag{6.2.26}$$

The wavefunction is

$$\Psi_{3/7}^{cft} = \mathcal{A}\left(\langle \prod_{i_3=1}^{M_3} V_3(z_{i_3}) \prod_{i_2=1}^{M_2} V_2(z_{i_2}) \prod_{i_1=1}^{M_1} V_1(z_{i_1}) \rangle \right) =$$
$$\mathcal{A}\left(\partial_3^2 (3-3)^3 \partial_2 (2-2)^3 (1-1)^3 (3-2)^2 (3-1)^2 (2-1)^2 \right) \tag{6.2.27}$$

where $M_1 = M_2 = M_3 = N/3$. We have three qh operators

$$H_1^3(z) = e^{\frac{i}{\sqrt{3}}\phi_1(z)} e^{\frac{-2i}{\sqrt{15}}\phi_2(z)} e^{\frac{-2i}{\sqrt{35}}\phi_3(z)}$$
$$H_2^3(z) = e^{\frac{3i}{\sqrt{15}}\phi_2(z)} e^{\frac{-2i}{\sqrt{35}}\phi_3(z)} \qquad H_3^3(z) = e^{\frac{5i}{\sqrt{35}}\phi_3(z)} \tag{6.2.28}$$

with charges $Q = \frac{1}{7}$ and monodromies giving statistical angles $\theta_{H_1} = \theta_{H_2} = \theta_{H_3} = \frac{5\pi}{7}$ and $\theta_{H_1 \leftrightarrow H_2} = \theta_{H_1 \leftrightarrow H_3} = \theta_{H_2 \leftrightarrow H_3} = -\frac{2\pi}{7}$.

6.2.3 Hierarchical wavefunctions

To go outside of the Jain sequence we have to consider condensates which are not maximum density, corresponding to partially filled CF Landau levels. While the calculations above bring the hierarchy of Haldane and Halperin to mind, using the quasilocal $\mathcal{P}$ it is possible to follow their model more strictly. As shown in [32] the resulting multiple integral for condensing qe's is

$$\Psi_{\nu_{n+1}}^{H/cft} = \prod_{i=1}^{M_n} \int d^2\eta_i \Phi_{n+1}^*(\eta_1, \ldots, \eta_{M_n}) \langle \prod_{j=1}^{M_n} \mathcal{P}(\overline{\eta}_j) \prod_{k=1}^{n} \prod_{l_k=1}^{N} V(z_{l_k}) \rangle \tag{6.2.29}$$

as one would expect, where Φ is the pseudowavefunction. Before computation the statistical phase is transferred from the monodromy to the Berry phase using additional boson fields as described earlier.

But by this use of CFT in the hierarchy wavefunction one ends up with deltafunctions making the integrals easy to solve, in contrast with the original hierarchical model. This has connection with the bosonization of the vertex operators, removing the fractional exponents from the integrands. We saw that the wavefunctions resulting from maximum density states are identical to those arising in the CF theory on the positive Jain series. Thus one may either evoke hierarchical qp condensing as above, or construct new electron vertex operators as we saw earlier, to reach a higher level n. Before examining states outside the Jain sequence we generalize the notation from the previous section following [50].

As the starting point we have the electron vertex operator $V_1(z) = e^{\gamma_1 i \phi_1(z)}$. At level n of the hierarchy there are $n-1$ additional operators iteratively constructed as $V_{\alpha+1}(z) =$

$\partial_z V_\alpha e^{\frac{-i}{\sqrt{\gamma_\alpha}}\phi_\alpha(z)} e^{i\gamma_{\alpha+1}\phi_{\alpha+1}(z)}$. The differential operator keeps the wavefunction nonzero, the first exponential gives the desired factor of $q-1$ in contracting two different operators as in section 6.2.2, while the second exponential introduces a new field making $V_{\alpha+1}$ fermionic. We define the exponential factors $\{s\}$ in the correlators:

$$\langle V_\alpha(z)V_\alpha(w)\rangle \sim (z-w)^{s_\alpha} \qquad \langle V_\alpha(z)V_\beta(w)\rangle \sim (z-w)^{s_{\alpha\beta}} = (z-w)^{s_{\beta\alpha}} \tag{6.2.30}$$

where the $\sim$ indicates that we only display the polynomials. Since the wavefunctions should be single-valued the s's must be integers, and odd such to have fermionic operators. From the definition of $V_{\alpha+1}$ we have

$$\begin{aligned}
\langle V_{\alpha+1}(z)V_{\alpha+1}(w)\rangle &\sim (z-w)^{s_\alpha}(z-w)^{-1}(z-w)^{-1}(z-w)^{-\frac{1}{\gamma_\alpha^2}}(z-w)^{\gamma_{\alpha+1}^2} \\
&\Longrightarrow \quad s_{\alpha+1} = s_\alpha - 2 + \gamma_\alpha^{-2} + \gamma_{\alpha+1}^2 \\
\text{assuming} \qquad \beta > \alpha \quad &: \quad \langle V_\alpha(z)V_\beta(w)\rangle = \langle V_\alpha(z)\partial_w \ldots \partial_w V_\alpha e^{\frac{-i}{\sqrt{\gamma_\alpha}}\phi_\alpha(w)}\mathcal{O}(\{\phi_{\lambda\neq\alpha}\})\rangle \\
&= (z-w)^{s_\alpha}(z-w)^{-1} \qquad \Longrightarrow \qquad s_{\alpha\beta} = s_\alpha - 1
\end{aligned} \tag{6.2.31}$$

From the second line of (6.2.31), since all s's are odd, $\gamma_\alpha^{-2} + \gamma_{\alpha+1}^2$ must be an even integer. We label this number $t_{\alpha+1}$ and write $t_1 \equiv \gamma_1^2$. Now the wavefunction is completely specified by the t's, of which the first is an odd integer and the rest are even. The wavefunctions at level n are given as

$$\Psi_{\nu_n}^{cft} = \mathcal{A}\langle \prod_{\alpha=1}^{n}\prod_{i_\alpha}^{M_\alpha} V_\alpha(z_{i_\alpha})\rangle \tag{6.2.32}$$

where the M_α's thus are the number of electrons in each subset and give the density of the state. As shown in [50] the filling factor is

$$\nu = \cfrac{1}{t_1 - \cfrac{1}{t_2 - \cfrac{1}{\ddots \cfrac{}{t_{n-1} - \cfrac{1}{t_n}}}}} \tag{6.2.33}$$

which we recognize from the hierarchy model. Figure 4.4 displays the filling fractions we have singled out (at level $n=1$ we have $\nu = 1/t_1$ while the subsequent p's are given by $t_{i>1}$).

A point we have not mentioned so far is that the CFT vertex operators can be written in different bases; we have used one among several. Using the state $\nu = 2/5$ as an example, instead of using the vertex operators in (6.2.1) and (6.2.14) we may for example use

$$V_1 = e^{i\frac{5}{\sqrt{10}}\phi_1 + \frac{i}{\sqrt{2}}\phi_2} \qquad V_2 = e^{i\frac{5}{\sqrt{10}}\phi_1 - \frac{i}{\sqrt{2}}\phi_2} \tag{6.2.34}$$

and modify the qp operators accordingly. In the basis we have employed up until now V_1 only contains ϕ_1 and each $V_{i>1}$ receives an additional field as the index rises. We will stick to this basis in the remainder. Next we use section 6.2.2 as guidance in finding an iterative scheme to identify the qh statistical angles corresponding to a particular set of t's.

Introducing constants $\lambda_i^{(j)} \in \mathbb{R}$ where $i \geq j$ we write the qh operators as

$$H_j = \prod_{k=0}^{n-j} e^{i\lambda_{n-k}^{(j)}\phi_{n-k}} \tag{6.2.35}$$

where H_j has $n-j+1$ factors containing the fields $\phi_n, \phi_{n-1}, \ldots, \phi_j$. Then the relation (6.2.20) specifies the λ's: When $i=j$, V_i and H_j have only one field in common, namely ϕ_i. To get unit exponent we then must have $\lambda_i^{(i)} = \gamma_i^{-1}$. Advancing one step $\lambda_{i+1}^{(i)}$ can be found demanding that correlators of adjacent qh operators equal 1 and using that H_i and V_{i+1} have the fields ϕ_i and ϕ_{i+1} in common. Finally the rest of the λ's are found using (6.2.20), iteratively descending in qh operator level and for each operator ascending in boson field level. The resulting expressions are

$$\lambda_i^{(j)} = \begin{cases} 1/\gamma_j & : \ j = i \\ \frac{1}{\gamma_i}\left(\gamma_j^{-2} - 1\right) & : \ i = j+1 \\ -\frac{1}{\gamma_i}\sum_{k=j}^{i-1} \lambda_k^{(j)}\left(\gamma_k - \gamma_k^{-1}\right) & : \ i > j+1 \end{cases} \qquad \gamma_i = \sqrt{t_i - \gamma_{i-1}^{-2}} \tag{6.2.36}$$

When we have the λ's the correlators of qh operators yield the following statistical angles:

$$\theta_i = \sum_{k=0}^{n-i} \left(\lambda_{n-k}^{(i)}\right)^2 \qquad \theta_{ij} = \sum_{k=0}^{n-j} \lambda_{n-k}^{(i)} \lambda_{n-k}^{(j)} \tag{6.2.37}$$

where $i<j$ indicates at what level the qh resides. While these expressions are derived in a specific basis the results should of course be general.

As an example of the above we consider the state $\nu = 7/17$, which can be described as a level $n=3$ condensate with $t_1=3$, $t_2=2$ and $t_3=4$. Using the γ-expressions and the first line in (7.4.6) we have $\lambda_1^{(1)} = 1/\sqrt{3}$, $\lambda_2^{(2)} = \sqrt{3/5}$ and $\lambda_3^{(3)} = \sqrt{5/17}$. The following line in (7.4.6) yields $\lambda_3^{(2)} = \frac{1}{\sqrt{\gamma_3}}(\gamma_2^{-2} - 1) = -2/\sqrt{85}$ and $\lambda_2^{(1)} = -2/\sqrt{15}$, while the last line gives $\lambda_3^{(1)} = -2/\sqrt{85}$. Using (6.2.37) we then have all the angles:

$$\begin{aligned} \theta_1 = \theta_2 &= \frac{11}{17} & \theta_3 &= \frac{5}{17} \\ \theta_{12} &= -\frac{6}{17} & \theta_{13} = \theta_{23} &= -\frac{2}{17} \end{aligned} \tag{6.2.38}$$

On the Jain sequence this iteration is simplified because $t_i = 2 \ \ \forall \ i > 1$. By calculating some γ's it seems that

$$\gamma_j = \sqrt{\frac{jt_1 - j + 1}{(j-1)t_1 - j + 2}} \tag{6.2.39}$$

We will prove this by induction. First we see that it is correct for γ_1, providing the starting point. Then we need to show that if it holds for j then it also does for $j+1$. We use the rightmost expression in (7.4.6):

$$\gamma_{j+1} = \sqrt{2 - \gamma_j^{-2}} = \sqrt{2 - \frac{(j-1)t_1 - j + 2}{jt_1 - j + 1}} = \sqrt{\frac{(j+1)t_1 - j}{jt_1 - j + 1}} \tag{6.2.40}$$

where in the second step we assume (6.2.39) to be correct. Since (6.2.40) is the same as the latter with $j \to j+1$ the result is proven.

Moreover, since all self-statistics angles are equal as are all relative angles we may use H_n and H_{n-1} (being the simplest qh operators in this basis) to find all the angles easily. We dub the angles θ_S and θ_R respectively, and after some algebra the result is

$$\theta_S = \theta_{H_n} = \frac{(n-1)t_1 - n + 2}{nt_1 - n + 1} \qquad \theta_R = \theta_{H_n \leftrightarrow H_{n-1}} = \frac{1 - \gamma_{n-1}^2}{2\gamma_{n-1}^2 - 1} \tag{6.2.41}$$

We take a closer look at a filling factor outside the Jain sequence, namely $\nu = 4/11$. By inspection we see that this may be created at level two with $t_1 = 3$ and $t_2 = 4$: $1/(3 - 1/4) = 4/11$. From (7.4.6) we have $\gamma_2 = \sqrt{t_2 - \gamma_1^{-2}} = 11/\sqrt{33}$, which leads to

$$\Psi_{4/11}^{II} = \mathcal{A}\langle \prod_{i_2=1}^{N/4} \partial_{i_2} e^{\frac{2i}{\sqrt{3}}\phi_1(z_{i_2})} e^{\frac{11i}{\sqrt{33}}\phi_2(z_{i_2})} \prod_{i_1=1}^{3N/4} e^{i\sqrt{3}\phi_1(z_{i_1})} \rangle = \mathcal{A}\left(\partial_2 (2-2)^5 (1-1)^3 (2-1)^2\right) \tag{6.2.42}$$

The numbers of electrons in each level $M_1 = 3N/4$ and $M_2 = N/4$ will be explained below. To compare this to the CF scheme we first go back to the second level of the Jain sequence. We rewrite (6.2.15) with $q = 3$, giving $\nu = 2/5$. In addition to the two indices i_1 and i_2 each running from 1 to $N/2$ we use an index i running from 1 to N. Then we have

$$\Psi_{2/5}^{II} = \mathcal{A}\left(\left(\prod_{i_2=1}^{N/2} \partial_{i_2} \prod_{i_2<j_2}^{N/2} (z_{i_2} - z_{j_2}) \prod_{i_1<j_1}^{N/2} (z_{i_1} - z_{j_1}) \right) \prod_{i<j}^{N} (z_i - z_j)^2 \right) \tag{6.2.43}$$

Now we see clearly that this corresponds to two filled CF Landau levels, with the last product explicitly performing the flux attachment (here with $p = 1$). As mentioned, what separates the Jain states from the more general is that in the former all electron levels are equally dense.

Next we return to the $\nu = 4/11$ wavefunction and rewrite it in the same way. We get:

$$\Psi_{4/11}^{cft} = \mathcal{A}\left(\left(\prod_{i_2=1}^{N/4} \partial_{i_2} \prod_{i_2<j_2}^{N/4} (z_{i_2} - z_{j_2})^3 \prod_{i_1<j_1}^{3N/4} (z_{i_1} - z_{j_1}) \right) \prod_{i<j}^{N} (z_i - z_j)^2 \right) \tag{6.2.44}$$

One of the subsets of electrons inside the first parenthesis now has an exponent of three, which corresponds to one of the CF Landau levels only being one-third full, in analogy with the exponent in the wavefunction of $\nu = 1/3$ taking us from the IQHE to the FQHE. Since the electrons in each level are assumed to reside in the same electron liquid droplet with homogeneous density there has to be thrice as many particles in the $n = 1$ level as in $n = 2$, hence the values of M_1 and M_2. It is clear that this wavefunction is not part of the Jain sequence.

We find the qh operators in the usual way, getting

$$H_1^2(z) = e^{\frac{i}{\sqrt{3}}\phi_1(z)} e^{\frac{-2i}{\sqrt{33}}\phi_2(z)} \qquad H_2^2(z) = e^{\frac{3i}{\sqrt{33}}\phi_2(z)} \tag{6.2.45}$$

As easily shown the current $J(z) = \frac{\partial_z \phi_1}{\sqrt{3}} + \frac{\partial_z \phi_2}{\sqrt{33}}$ gives both electron operators unit U(1) charge (and thereby zero local charge in the electron liquid). H_2^2 has the normal U(1) charge $Q = 1/11$ giving $Q_E = e/11$. But we see that H_1^2 has an electric charge $Q_E = 3e/11$. This is in keeping

with the fact that the electrons in the $n = 1$ level are three times more dense than in the $n = 2$ level, so that a (partial) vortex acting on the former will have a vorticity three times larger than on the latter. Therefore the fundamental qh's at filling factors away from the Jain sequence may be inequivalent. Looking at the statistics angles these are also inequivalent; we obtain $\theta_{H_1} = \frac{5\pi}{11}$, $\theta_{H_2} = \frac{3\pi}{11}$ and $\theta_{H_1 \leftrightarrow H_2} = \frac{-2\pi}{11}$.

6.2.4 Quasielectron statistics

As mentioned the question of qe statistics in CFT is difficult. In both the method using P and the one using $\mathcal{P}$ the statistics is moved from the monodromy to the Berry phase, which has not been calculated in closed form. However both constructions use H_n^{-1} as the starting point. In analogy with the expectation that the complete qh statistics angle initially resides in the monodromy of H we may speculate whether the same holds for the qe. By inspection of H_n^{-1} we see that if this is the case the qe angle equals the one from the qh with the highest index n:

$$\theta_{qe} = \theta_{H_n^{-1}} = \gamma_n^{-2} \tag{6.2.46}$$

since the minus in the exponent has no effect on the angle.

6.3 Wen's classification scheme

In his work on CS theory [43] Wen has introduced some useful quantities that enable us to compare results from CFT and the other models with those from CS theory (simply quoting the latter in this account, all results below are from [47] and [50]). The CS theory for Abelian QHE fluids may be characterized by a matrix K, the vector $\mathbf{t}$ and n vectors $\mathbf{l}^{(\alpha)}$ forming a matrix $L_{\alpha\beta} = l_\alpha^{(\beta)}$. These are not unique but can be expressed in different bases. In what is called the symmetric basis we always have $\mathbf{t} = (1, \ldots, 1)$ and $L = \mathbb{1}$ so that the $\mathbf{l}$'s are orthonormal. The Lagrangian and filling fraction are

$$\mathcal{L} = -\frac{1}{4\pi} K_{\alpha\beta} a_\mu^\alpha \partial_\nu a_\lambda^\beta \epsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu t_\alpha a_\lambda^\alpha \epsilon^{\mu\nu\lambda} \qquad \nu = \mathbf{t}^T K^{-1} \mathbf{t} \tag{6.3.1}$$

where there are n gauge fields a_μ^α (α is a hierarchy level index while μ is a spacetime index) and $\epsilon^{\mu\nu\lambda}$ is the antisymmetric Levi-Civita tensor. (6.3.1) implies that the dimensions of K, $\mathbf{t}$ and $\mathbf{l}^\alpha$ are equal to the hierarchy level.

The elements of K give the exponents in the CFT wavefunction of the relative electron coordinates at the different levels (while the number of differentials is always $n - 1$). For example at the second level the wavefunction is given by

$$\Psi = \mathcal{A}\left(\partial_2 (1-1)^{K_{11}} (2-2)^{K_{22}} (1-2)^{K_{12}}\right) \tag{6.3.2}$$

If we then introduce a vector $\mathbf{Q} = (Q_1, \ldots, Q_n)$ containing the qh U(1) charges and a matrix $\theta_{\alpha\beta}$ containing the qh statistisc angles (the self-statistics thus residing on the diagonal) Wen's theory gives

$$\mathbf{Q} = \mathbf{t}^T K^{-1} L \qquad [\theta_{\alpha\beta}] = \pi L^T K^{-1} L \tag{6.3.3}$$

We note that in the symmetric basis the statistics angles can simply be read off from the inverse of K. As an example of the above results we look at the $\nu = 2/5$ system. In the symmetric basis the K matrix is [46]

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{Q} = [1\ 1]\frac{1}{5}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\mathbb{1} = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}$$
$$\theta = \pi\mathbb{1}\frac{1}{5}\begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix}\mathbb{1} = \begin{bmatrix} 3\pi/5 & -2\pi/11 \\ -2\pi/11 & 3\pi/5 \end{bmatrix} \tag{6.3.4}$$

We recognize the charge and statistics values we found using the other models. In summary Wen's classification in the symmetric basis supplies most relevant quantities of a FQHE system (filling fraction, wave function, qh charge and statistics) in a single matrix K. There are no known discrepancies between the results for the statistics angle from CFT and CS theory.

Chapter 7

The statistics angle: Comparison of the different approaches

In the preceding chapters we have examined different models explaining the QHE with emphasis on the exchange statistics angle θ. This angle is only defined mod 2π, since a factor $\exp(2\pi i)$ is trivial. In this chapter we will compare the predictions concerning θ, examine an argument by Su in light of results from CFT and list the expected angles for some of the QHE filling fractions. But first some comments concerning the angle.

7.1 General considerations

An important question concerning the qp statistical angle is whether it is a robust quantity, that is whether it is unchanged after an adiabatic change of the Hamiltonian preserving the energy gap. The angles of fermions and bosons are certainly robust, and the fact that they are supposedly topological points towards this conclusion for the qp angles also. One caveat is that the qp's are not point particles and therefore not ideal anyons; the exchange angle is only well defined when the qp's do not approach too closely. In section 5.2.1 we saw that the numerical calculations of qp Berry phases are sensitive to details in the setup. In [37] Jain argues that this means that the statistical angle is more fragile than expected. The ultimate judge of this question is experiment - we will assume that θ is robust and see what we can say about it.

In the original three-dimensional derivation of particle statistics there is an assumption that the exchanged particles are identical. Similarly, in our presentation of anyon statistics in chapter 3 we argued that the singularity in the configuration space is due to identical particles. However in chapter 6 we saw that qp's which are not identical in the usual sense also are expected to gain phase factors when interchanged. This is what we referred to as relative statistics, as opposed to self-statistics. It arises in connection between unequal qh's at a given filling factor. The question whether these are distinguishable or indistinguishable in the traditional sense is difficult but since they have different charge and self-statistics the first option is most evident. Then the required singularity in the configuration space has to arise in a somewhat different manner. This is also hinted at by the fact that θ_S is positive while the θ_R's are negative (or equivalently $> \pi$) in the cases we have encountered.

As we saw in section 6.2.3, using CFT we may create composites of qh operators with $Q = e$, the charge of a hole. For example at $\nu = 2/5$ we may construct holes with $5 \times H_1$,

$5 \times H_2$ or a mixture of these. According to CFT all these holes have fermionic self-statistics, a conclusion also supported by clustering arguments in section 7.2. But it turns out that the relative statistics are bosonic. This indicates that either the exchange statistics between these holes are different than the usual, or that they are regular but non-identical holes. We will refrain from further speculations about these matters and simply present the results concerning relative statistics.

In the following we will sometimes refer to elementary or fundamental qp's. These are the smallest qp excitations at a given filling factor - there are possibilities of less elementary ones, e.g. Laughlin-type vortex qp's outside the Laughlin sequence. This definition is not a strict one however. A more pragmatic one would be: the fundamental qh's are those fulfilling the CFT relation (6.2.20) and the fundamental qe is the antiparticle of the qh in the highest level.

7.1.1 The hierarchy and composite fermions revisited

When looking at the hierarchy and CF models we arrived at a single value for the qh self-statistics angles even though we expect several distinct qh's to exist. In the case of the hierarchy this is to be expected, because the qp's referred to are the ones forming the gapped condensate, that is the particles in the highest level. This also explains why the qh and qe angles are equal, as we saw in the chapter 6 that they indeed are when it comes to the qh in the highest level.

Using the CF scheme we derived the qh statistics in a roundabout way, through the qe statistics and appealing to clustering arguments. Therefore it is no surprise that we could only say something about qh's with charge of equal magnitude as the qe. As we saw in chapter 6, this applies to all the qh's on the Jain sequence.

7.2 Clustering arguments

In sections 4.2.3 and 5.2.1 we employed clustering arguments, with the main observations (4.2.22) and (4.2.23). This was inspired by Su's article [26] in which he uses such arguments to uniquely specify the statistics of the elementary qh's (eqh) in the hierarchy, assuming that there is only one kind of qh at each ν. Here we will look at his arguments without this assumption and also examine qe's.

To be able to compare predictions from clustering to those of the other models we will try to avoid incorporating assumptions from these in the argumentation. There are however three such that we will use: Firstly, as mentioned in section 4.2.2 we should always have the possibility of creating a Laughlin qp using the solenoid argument, with charges $Q_{Lqp} = \pm\nu e$ and statistics angles $\theta_{Lqh} = \theta_{Lqe} = \nu\pi$. To create such a qh at level n we need one each of the n potentially different eqh's, as outlined in sections 5.2 and 6.2.2. Secondly we take the elementary qp charges $Q_{eqp} = \pm re/q$ where $r \in \mathbb{N}$ and the filling factor is $\nu = p/q$ as input.

Thirdly we assume that various composites of qh's with a total charge of unity have the same relative statistics when moved around a Laughlin qh. This is inspired by the self-consistent assumption in Su's article. In our context it cannot be shown to be self-consistent (because Laughlin's qp's are not generally elementary), but it is supported by explicit calculations using CFT and in line with the connection between charge and statistics cf. section 7.3. In the next section we look at the clustering arguments in the context of the Jain series where from (5.2.8) we know that $Q_{eqp} = \pm 1/q$, and then we look at the general case.

7.2.1 The Jain sequence

Su considers a cluster $C = A + B$ where A consists of n_1 Laughlin qh's and B of n_2 holes (not qh's), while $n_1 p + n_2 q = 1$ (which always can be satisfied due to Bézout's identity and the fact that p and q are relatively prime). This ensures that the composites have charge $1/q$, which is the asssumed eqh charge in the article. However this means that either n_1 or n_2 must be below zero, implying a negative amount of qh's or holes. This is unproblematic for the charge considerations when there is only one kind of eqh because the end result is still a positive amount of eqh's. But the following arguments dealing with statistics may be ambiguous because the statistics angle of a negative amount of particles is used in the argumentation.

One may argue that a negative amount of qh's or holes refers to qe's or electrons. In this spirit and to avoid the question we consider a cluster $C = A + B$ where A consists of n_1 Laughlin qh's and B of n_2 electrons instead of holes, and demand

$$n_1 p - n_2 q = \pm 1 \tag{7.2.1}$$

where the minus sign will be used to look at elementary quasielectrons (eqe). Bézout's identity does not guarantee that (7.2.1) can always be satisfied but with trial and error we find a solution in all cases considered. It seems reasonable that an elementary quasiparticle can to some extent be modelled by Laughlin qh's and electrons; creating a composite that is effectively either a smaller qh or a "partial electron."

To find the charge of C we simply add the charges of the individual constituents. For the statistics angle of C with itself we have $\theta_C = \theta_A + \theta_B + 2\theta_{A\leftrightarrow B}$. This is self-statistics as opposed to relative statistics, hence the single index. Next we assume that moving an electron (a fermion) around a Laughlin qh gives a phase $p\pi \pmod{2\pi}$ with integer p [1]. This means that $\theta_C = \theta_A + \theta_B \pmod{2\pi}$. We have $\theta_B = n_2^2\theta_e = n_2^2\pi$ [2], and using (4.2.22) we find $\theta_A = n_1^2\theta_{Lqh} = n_1^2\nu\pi$. This yields an expression for the statistics of C, which together with the charge Q_C is

$$\begin{aligned} \theta_C &= n_1^2\frac{p}{q}\pi + n_2^2\pi = \frac{\pi}{q}(pn_1^2 + qn_2^2) \\ Q_C &= n_1\frac{p}{q}e - n_2 e = \frac{e}{q}(pn_1 - qn_2) \end{aligned} \tag{7.2.2}$$

From (7.2.1) this gives $Q_C = \pm e/q$ so that C has the charge of the elementary qp's (assuming $r = 1$ for now). There are however different choices of n_1 and n_2 that satisfy the above relation. If we change n_i by Δ_i and require the equality (7.2.1) to still hold we get

$$(n_1 + \Delta_1)p + (n_2 + \Delta_2)q = \pm 1 = n_1 p + n_2 q \quad \Longrightarrow \quad \Delta_2 = \frac{p}{q}\Delta_1 \tag{7.2.3}$$

producing the following change $\Delta\theta_C$ in the angle:

$$\begin{aligned} \theta_C \to \theta_C' &= \frac{\pi}{q}\left((n_1 + \Delta_1)^2 p + (n_2 + \Delta_2)^2 q\right) \\ &= \theta_C + \frac{\Delta_1^2}{q^2}p(p+q)\pi + \frac{\Delta_1}{q}2p(n_1 + n_2)\pi \equiv \theta_C + \Delta\theta_C \end{aligned} \tag{7.2.4}$$

[1] In [26] Su uses a hole instead of an electron and is able to show that the equivalent assumption is self-consistent within the argumentation. With an electron in place of the hole we are not able to do this but the assumption is probable and supported by the calculation of Arovas et al and CFT.

[2] Exchanging two B-composites with each other entails interchanging n_2 electrons with n_2 electrons, giving a total of n_2^2 exchanges.

But because n_2 must be an integer while p and q are relatively prime, (7.2.3) implies that $\Delta_1 = mq$ for some $m \in \mathbb{N}$ (p and q have no common factors but this is needed to obtain an integer, meaning that Δ_1 must contain such factors. Since Δ_1 and q are integer m must also be). For odd q we know $p(p+q)$ is even, therefore (7.2.4) gives

$$\Delta\theta_C = m^2 p(p+q)\pi + m2p(n_1+n_2)\pi = 0 \mod 2\pi \tag{7.2.5}$$

The above means that there is a one-to-one correspondence between charge and self-statistics in the case of the eqp's, and that they can for some purposes be modelled with C. Assuming the charge of a qp is $\pm 1/q$ we can find two integers n_1 and n_2 that obey (7.2.1) and use these with (7.2.2) to predict eqp's self-statistics angle. We have nowhere assumed that the eqh's are equivalent at a given hierarchy level.

As a consistency check we try to model an electron with θ_C. To obtain a charge $Q_C = -e$ we can choose $n_1 = q$ and $n_2 = p+1$. The result is fermionic as expected:

$$\theta_C = \frac{\pi}{q}\left(pq^2 + q(p+1)^2\right) = \pi p(p+q) + \pi \mod 2\pi = \pi \mod 2\pi \tag{7.2.6}$$

Since there is only one qe at a given ν with statistics given by 7.2.2 we have exhausted the possibilities concerning qe's and electrons. In the following we concentrate on qh's and holes.

Below we will look at different composites of qh's and see what we can say about the statistics when we move them around each other, using the assumptions listed in section 7.2. We then deduce some constraints on the statistics of the elementary qh's on the Jain sequence at CF LL or equivalently hierarchy level n. Firstly, since they all have the same charge at a given level the charge-statistics correspondence gives

$$\theta_{qh_i} = \theta_{qh_j} \quad \forall\ i,j \tag{7.2.7}$$

where qh_i indicates a qh residing at level i. For example at $\nu = 2/5$, where we have the CFT eqh operators H_1^2 and H_2^2, these must satisfy $\theta_{H_1} = \theta_{H_2}$. As in section 6.2.3 we therefore name the self-statistics angles in the Jain series θ_S.

A thought we will pursue in a later section, and which supports among other things the third assumption in section 7.2, is that charge and statistics are closely related in the QHE. Logically this should mean that a hole h created from q identical eqh's and thus with charge e has fermionic self-statistics, although it is not necessarily a normal hole as an antiparticle of the electron (see the comments below (6.2.23)). We confirm this:

$$\theta_h = q^2\theta_C = \pi(n_1^2 pq + n_2^2 q^2) = \pi(1 + n_1^2 p(q-p)) \mod 2\pi = \pi \mod 2\pi \tag{7.2.8}$$

where we have used $n_2^2 q^2 = 1 + 2n_1 n_2 qp - n_1^2 p^2$ from (7.2.1).

Next we picture some other kinds of holes. The first one we create from $q-1$ eqh's of one type plus a different eqh, for the second we use $q-2$ eqh's of one type and 2 of another, and so on. Then we utilize our third assumption; namely that all these composites have the same relative statistics when moved around a Laughlin qh (figure 7.1). This means that if we for

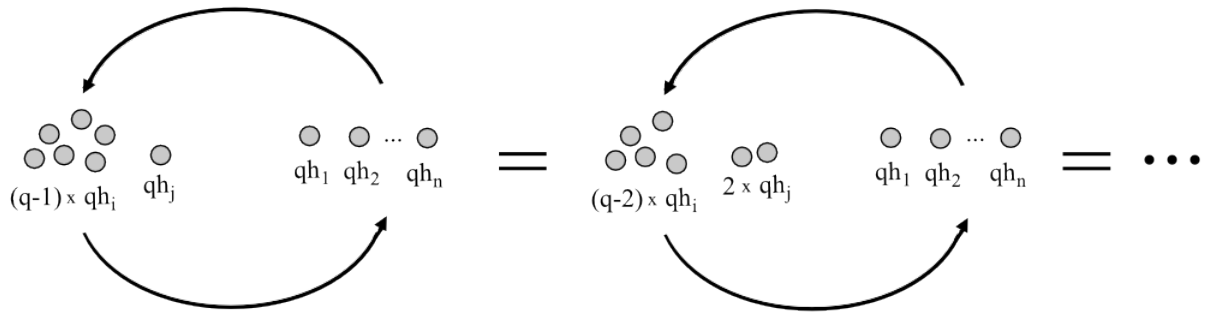

Figure 7.1: Exchanging different holes with Laughlin quasiholes

example single out qh_1 and qh_2 we have

$$(q-1)\theta_1+\theta_2+(q-1)\sum_{i\neq 1}^{n}\theta_{1i}+\sum_{i\neq 2}^{n}\theta_{2i}=(q-2)\theta_1+2\theta_2+(q-2)\sum_{i\neq 1}^{n}\theta_{1i}+2\sum_{i\neq 2}^{n}\theta_{2i}$$
$$=\ldots=\theta_1+(q-1)\theta_2+\sum_{i\neq 1}^{n}\theta_{1i}+(q-1)\sum_{i\neq 2}^{n}\theta_{2i}$$
$$\Longrightarrow\qquad (q-1)\sum_{i\neq 1,2}^{n}\theta_{1i}+\sum_{i\neq 1,2}^{n}\theta_{2i}=\ldots=\sum_{i\neq 1,2}^{n}\theta_{1i}+(q-1)\sum_{i\neq 1,2}^{n}\theta_{2i}\mod 2\pi \tag{7.2.9}$$

where we have used that all the self statistics angles are equal and $\theta_{12}=\theta_{21}$. The mod 2π applies to all the equalities above and we assume $n>2$ (the argumentation in this paragraph is only useful in those cases). There are $\frac{n(n-1)}{2}-2$ unknowns and $q-1$ equations. By inspection we see that the number of equations is then greater than the number of unknowns for all filling fractions. Thus the solution, if it exists, is unique. We immediately see that one solution is

$$\theta_{ij}=\theta_{kl}\mod 2\pi\qquad i\neq k\quad j\neq l\quad i,j,k,l\neq 1,2 \tag{7.2.10}$$

since then all the equations say $q(n-2)\theta=q(n-2)\theta$. The mod does not compromise the result: An angle can be changed by 2π and the equations still hold, but the angles are only defined mod 2π in any case.

By singling out other pairwise angles we can obtain analogous relations as in (7.2.10) for all of them. This means that we must have

$$\theta_{ij}=\theta_{kl}\mod 2\pi\qquad i\neq k\quad j\neq l \tag{7.2.11}$$

on the Jain sequence. Therefore we do as when using CFT and dub all the relative angles on a given filling factor θ_R.

Lastly we consider a Laughlin qh with statistics angle $\theta_{Lqh}=\nu$. We compose this from n eqh's, one of each kind, and move one around the other (see figure 7.2). Using (7.2.7) and (7.2.11) we get (since there are $n(n-1)$ relative statistical angles at level n)

$$\theta_{Lqh}=\nu \bmod 2\pi=n\theta_S+n(n-1)\theta_R$$
$$\Longrightarrow\quad \theta_R=\frac{\pi}{n(n-1)}(\nu\pi-n\theta_S)\mod\frac{2\pi}{n(n-1)} \tag{7.2.12}$$

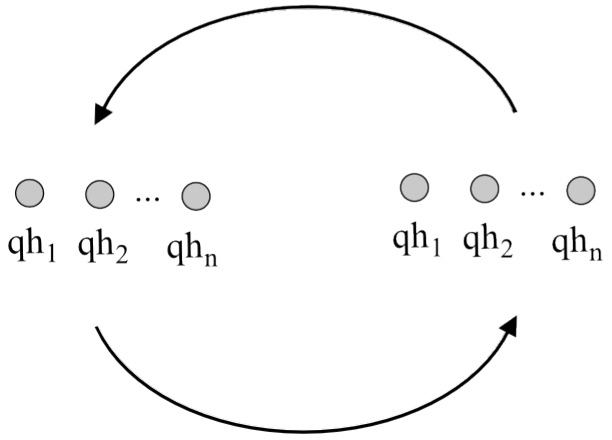

Figure 7.2: Interchange of Laughlin quasiholes

Since there is no θ_R at $n = 1$ the associated diverging expression should be disregarded.

The physical angles on the other hand are defined mod 2π. On the Jain sequence we thus cannot deduce the relative statistics uniquely but only reduce their possible values to a set of angles. This set grows bigger with larger n. What we can say about relative statistics using clustering arguments is therefore to some extent limited.

To recap, with the qp charges as input we have found the following conditions on θ in the Jain sequence: The self-statistics angles are all equal and given by the charge while the relative statistics angles are also equal and fixed mod $2\pi/n(n-1)$ by (7.2.12). Considering other composites and exchange situations does not seem to add any new information. We look at an example:

At $\nu = 3/7$ we know from other models that the eqp charge is $Q = 1/7$. To fulfil (7.2.2) we may use $n_1 = 2$ and $n_2 = 1$ for the qe, and $n_1 = 5$ and $n_2 = 2$ for the qh's. The self-statistics angles are then

$$\theta_S^{qe} = \frac{\pi}{7}\left(3 \cdot 2^2 + 7\right) = \frac{5\pi}{7} \mod 2\pi \qquad \theta_S^{qh} = \frac{\pi}{7}\left(3 \cdot 5^2 + 7 \cdot 2^2\right) = \frac{5\pi}{7} \mod 2\pi \quad (7.2.13)$$

while the qh relative statistics angles are constrained by

$$\theta_R = \frac{\pi}{6}\left(\frac{3}{7}\pi - 3\frac{5\pi}{7}\right) \mod \frac{\pi}{3} = -\frac{2\pi}{7} \mod \frac{\pi}{3} \quad (7.2.14)$$

We note that this may look a little strange when considering that all other models give statistics angles $\theta = \pm\frac{re}{q}$ where $\nu = p/q$ and r is an integer, while the above expression allows for other denominators. But apparently there are no general arguments to support this limit on the fractions. Using it as an assumption we could confine θ_R more strictly, but we will refrain from this.

7.2.2 Outside the Jain sequence

Clustering in the context of more general filling factors, where the eqp charge is not necessarily $Q = \pm 1/q$, is more difficult to handle but with similar results. We start out with an analogous cluster as in the beginning of section 7.2.1: $C = A + B$ with A consisting of n_1^i Laughlin qh's

Particle	n_1	n_2
qe	12	5
qh_1	15	6
qh_2	15	6
qh_3	5	2

Table 7.1: Values of n_1 and n_2 giving the correct charge of the $\nu = 7/17$ eqp's

and B of n_2^i holes. But now we require *each* qp charge $Q = r^i e/q$ as input. i is the hierarchy index and r can be negative or positive according to whether we are considering qe's or qh's. We then look at each eqp separately with the requirement

$$n_1^i p - n_2^i q = r^i \tag{7.2.15}$$

so that the clusters have the charges of the eqp's. Once again there is no guarantee that these have solutions, but trial and error show that they do. With these two integers we may again find the self-statistics angle with (7.2.2). The proof that the connection is one-to-one is completely analogous to the steps in (7.2.3) through (7.2.5).

But to manufacture a hole or an electron we can no longer just use q qh's or qe's respectively; the charge of the resulting composite would have absolute value $|Q| > e$. Naively we could imagine doing this and ending up with a number of holes or electrons but since these are supposed to be fermionic this would violate the Pauli exclusion principle and thus gives unphysical results. Therefore we must instead treat each filling factor separately to follow steps corresponding to those in equations (7.2.6) through (7.2.12); there are no general expressions like we found on the Jain sequence. As illustration we look at the example $\nu = 7/17$, and list the rest of the results in section 7.4.

From section 6.2.3 we know that the eqp charges are $Q_{qe} = -e/17$, $Q_{qh_1} = Q_{qh_2} = 3e/17$ and $Q_{qh_3} = e/17$. In table 7.1 are the values of n_1 and n_2 that can be used to model these particles with (7.2.15), giving the following statistics angles:

$$\begin{aligned} \theta_{qe} &= \frac{\pi}{17}\left(7 \cdot 12^2 + 17 \cdot 5^2\right) = \frac{5\pi}{17} \mod 2\pi \\ \theta_{qh_1} = \theta_{qh_2} &= \frac{11\pi}{17} \mod 2\pi \qquad \theta_{qh_3} = \frac{5\pi}{17} \mod 2\pi \end{aligned} \tag{7.2.16}$$

Going from the charge, we can use 17 qe's to describe an electron and 17 of the third qh to describe a hole. We check that this is consistent:

$$\theta_e = 17^2\theta_{qe} = \pi \mod 2\pi \qquad \theta_h = 17^2\theta_{qh_3} = \pi \mod 2\pi \tag{7.2.17}$$

We then look for qh composites that might be useful to exchange with Laughlin qh's. The following combinations also give a particle of charge $Q = e$: $1 \times qh_1 + 14 \times qh_3$ and $1 \times qh_2 + 14 \times qh_3$. If we exchange both of these around a Laughlin qh and demand that the results are equal we get, after some algebra,

$$\frac{13\pi}{17} + \theta_{12} + 15\theta_{13} + 14\theta_{23} = \frac{13\pi}{17} + \theta_{12} + 14\theta_{13} + 15\theta_{23} \mod 2\pi \quad \Longrightarrow \quad \theta_{13} = \theta_{23} \mod 2\pi \tag{7.2.18}$$

We then go back to the composite $17 \times qh_3$ and move it around a Lqh, demanding that the result is equal to the leftmost expression in (7.2.18) and using $\theta_{13} = \theta_{23}$. The result is

$$\theta_{12} = \frac{4\pi}{17} + 5\theta_{13} \mod 2\pi \tag{7.2.19}$$

Lastly we picture moving one Laughlin qh around another, giving statistics angle $\nu\pi$, and use the results in (7.2.16), (7.2.18) and (7.2.19). We then have

$$\begin{aligned} \frac{7\pi}{17} \mod 2\pi = \theta_1 + \theta_2 + \theta_3 + 2(\theta_{12} + \theta_{13} + \theta_{23}) = \frac{35\pi}{17} + 14\theta_{13} \\ \implies \quad \theta_{13} = \theta_{23} = -\frac{2\pi}{17} \mod \frac{\pi}{7} \quad \implies \quad \theta_{12} = -\frac{6\pi}{17} \mod \frac{\pi}{7} \end{aligned} \tag{7.2.20}$$

With that we have the self-statistics angles and have limited the relative angles to a set of fractions. As before the set contains fractions with denominators other than q, so that if we assume these to be unphysical we could again restrict the set to within mod π. No other clustering processes in the same vein as the ones used above give any further results. All filling fractions off the Jain sequence can be handled in a similar way, the results for some of these are in tables 7.2 through 7.6.

7.3 Q and Θ, an observation

Charge is a less ephemeral quantity than statistics, and fractional charge has been observed in experiment in contrast to fractional statistics. But one trend that most of the previous sections brings to mind is that there is a close connection between Q and θ. Following is a list of factors that points to this assertion:

In the paper by Arovas et al the same Berry phase calculation is used to derive both the charge and the statistics. This phase contains a contribution from the movement of charge in a magnetic field and one due to exchange statistics. Halperin's version of the hierarchy contains interdependent equations leading to Q and θ. In the CF scheme they both arise from the vorticity of the CF quasiparticles, and in CFT the exponents of qp operators fixes them both. Finally, in the previous section we saw that when the qp charge is given the self-statistics are uniquely determined.

There is another argument by Kivelson and Rŏcek [51] that links the two quantities in the case of a Laughlin qh. As observed, transporting one qh around another yields an Aharonov-Bohm phase factor $\exp(2\pi/q)$ on the wavefunction; since the density goes to zero in the centre and thus can be thought of as a unit flux tube, while the qh charge is e/q. But due to the Byers-Yang theorem [52] the wavefunction must be single valued if the flux through the system is a whole number of flux quanta. Thus the fractional exponent from the AB phase must be cancelled by another phase - namely a fractional statistical phase (equal to half the AB phase since transporting a qh completely around another correspond to two interchanges).

Using CFT we made different composites from qh operators with charge $Q = e$ and saw that they had fermionic statistics like holes. In section 7.2 we assumed that all such hole-like particles have the same statistics when moved around a Lqh, which is supported by the θ-Q connection and CFT. The subsequent steps leading up to (7.2.12) show that if this is the case the relative statistics also are constrained by the charges. Similar comments apply to section 7.2.2.

The tables 7.2, 7.4 and 7.5 confirm the correlation between Q and θ. Looking at for example $\nu = 7/17$, qh_1 and qh_2 have the same charge while qh_2's is different. In conjunction with this θ_1 equals θ_2 but not θ_3, and θ_{13} equals θ_{23} but not θ_{12} (the first two relative angles concern particle pairs with equal charge allotment).

7.4 Summary

In this final section we revisit each model considered and quote the results for the statistics angles. We then use these formulas and algorithms to compute the angles for our chosen filling fractions (all of which are expected to have qp's with Abelian statistics, see section 4.3) and list the results, both for the charges and statistics angles since the former are needed to compute some of the latter.

Laughlin's trial wavefunctions were the first descriptions of FQHE qp systems. They deal with filling factors $\nu = 1/q$, the Laughlin series, and the qp charges are $Q = \pm\nu e$. Following the calculation of Arovas et al with an alteration from Hansson et al plus Kjønsberg and Leinaas we found the qh statistics angles on the Laughlin sequence. Furthermore, with clustering arguments similar to those of Su we showed that the qe angles should be the same:

$$\theta_{Lqh} = \theta_{Lqe} = \nu\pi \tag{7.4.1}$$

The hierarchy construction of Haldane and Halperin was the next model in line. Using this one may describe filling factors of any fraction but the QHE stability falls with higher denominators. Following Halperin we showed that the following iterative equations give the expected self-statistics angles of the qe's and the qh's at the highest hierarchy level for a given filling factor ν_n:

$$\begin{aligned} s_n &= 2p_s - \frac{\alpha_n}{s_{n-1}} & \zeta_n &= \frac{\alpha_n\zeta_{n-1}}{s_n} \\ \nu_n &= \nu_{n-1} + \frac{\alpha_n|\zeta_{n-1}|\zeta_{n-1}}{s_n} & \theta_{qh} &= \theta_{qe} = \frac{\pi}{s_n} \end{aligned} \tag{7.4.2}$$

with the initial configuration $\zeta_0 = s_0 = \alpha_1 = 1$ and $\nu_0 = 0$, and qp charges $Q = -\alpha_{n+1}\zeta_n e$.

Using the composite fermion model we may produce wavefunctions for $\nu = p/q = \frac{k}{2pk+1}$, called the positive Jain series. The charges of qp's are $\pm e/q$ and all the self-statistics angles are

$$\theta_S = \pi\frac{2p(k-1)+1}{2pk+1} \tag{7.4.3}$$

With conformal field theory we can create wavefunctions for all filling factors associated with condensing qe's, in a way apparently unifying the hierarchical and composite fermion models. The charge and statistics angles of qp's are coded in operators, with the caveat that the qe statistics angles are not strictly derived but shown to probably equal that of the qh in the highest level. With the filling factor

$$\nu_n = \cfrac{1}{t_1 - \cfrac{1}{t_2 - \cfrac{1}{\ddots \cfrac{}{t_{n-1} - \frac{1}{t_n}}}}} \tag{7.4.4}$$

the self - and relative statistics angles of qh's are generally

$$\theta_i = \pi \sum_{k=0}^{n-i} \left(\lambda_{n-k}^{(i)}\right)^2 \qquad\qquad \theta_{ij} = \pi \sum_{k=0}^{n-j} \lambda_{n-k}^{(i)} \lambda_{n-k}^{(j)} \tag{7.4.5}$$

where $i < j$ and with the iterative equations

$$\lambda_i^{(j)} = \begin{cases} 1/\gamma_j & : \ j = i \\ \frac{1}{\gamma_i}\left(\gamma_j^{-2} - 1\right) & : \ i = j+1 \\ -\frac{1}{\gamma_i} \sum_{k=j}^{i-1} \lambda_k^{(j)} \left(\gamma_k - \gamma_k^{-1}\right) & : \ i > j+1 \end{cases} \qquad \gamma_i = \sqrt{t_i - \gamma_{i-1}^{-2}} \tag{7.4.6}$$

where the qe angle is arguably equal to that of the n'th qh. On the positive Jain sequence the expressions are simplified:

$$\theta_S = \pi \frac{(n-1)t_1 - n + 2}{nt_1 - n + 1} \qquad\qquad \theta_R = \pi \frac{1 - \gamma_{n-1}^2}{2\gamma_{n-1}^2 - 1} \tag{7.4.7}$$

where

$$\gamma_j = \sqrt{\frac{jt_1 - j + 1}{(j-1)t_1 - j + 2}} \tag{7.4.8}$$

The above expressions are not very straightforward however. In many cases it may be easier to go through the motions and write down the actual qp vertex operators as in chapter 6, from which the statistics can be found by inspection. When it comes to charge we have not derived any closed expressions, and to fill in the tables with charge we examine vertex operators.

The statistics angles can also be calculated using CS theory, as we saw in section 6.3. The results are equal to those of CFT.

Finally we have used clustering arguments following Su with some additional assumptions. We assume the filling factor is $\nu = p/q$, the qe charge is $-r^n e/q$ and qh charges are $r^i e/q$ where $i = 1, \ldots, n$ is the hierarchy/CF LL index. If we then find two integers so that $n_1^i p - n_2^i q = \pm r^i$ the self-statistics angles are

$$\theta_i = \frac{\pi}{q}\left(n_1^{i2} p + n_2^{i2} q\right) \tag{7.4.9}$$

On the Jain sequence, with $n_1 p - n_2 q = \pm 1$, the angles are

$$\theta_S = \frac{\pi}{q}\left(n_1^2 p + n_2^2 q\right) \qquad\qquad \theta_R = \frac{\pi}{n(n-1)}\left(\nu - n\theta_S\right) \mod \frac{2\pi}{n(n-1)} \tag{7.4.10}$$

while the relative angles outside the Jain sequence can be found with algebraic methods that we were not able to state as general formulas.

In tables 7.2 through 7.6 results for qp charges and statistics angles are listed for the filling factors meeting the criteria adopted in section 4.3. The dashes in the CF column indicate that the model does not predict any statistics angles outside the Jain sequence. Similarly the H column lists only one qh angle even outside the Jain sequence since the equations only concern the qh in the highest level. The column with clustering argument results also shows how much

	Hierarchy	CF	CFT
ν	Q_E/e	Q_E/e	Q_E/e
1/q	1/q	1/q	1/q
2/5	1/5	1/5	1/5
2/9	1/9	1/9	1/9
2/13	1/13	1/13	1/13
2/17	1/17	1/17	1/17
4/11	1/11	-	$Q_1 : 3/11 \;\; Q_2 : 1/11$
4/19	1/19	-	$Q_1 : 3/19 \;\; Q_2 : 1/19$
3/7	1/7	1/7	1/7
3/13	1/13	1/13	1/13
3/19	1/19	1/19	1/19
7/17	1/17	-	$Q_1, Q_2 : 3/17 \;\; Q_3 : 1/17$
7/19	1/19	-	$Q_1 : 5/19 \;\; Q_2, Q_3 : 1/19$
4/9	1/9	1/9	1/9
4/17	1/17	1/17	1/17
5/11	1/11	1/11	1/11
6/13	1/13	1/13	1/13

Table 7.2: Electric charge of elementary qh's

the angles are constrained by giving the modulo angle. Table 7.5 has just two columns since only CFT and clustering arguments predict relative angles.

Looking at the tables the first thing that strikes us is that all the results from the different approaches agree with each other. If any models had disagreed it would probably have been pointed out earlier. But a comprehensive list, also containing clustering arguments in new contexts, shows the coherence in an obvious manner.

One might question the extent of independence between these models and whether the results agree because the models, or at least some of them, are closely related or employ data from each other (like the clustering arguments and CFT). But at least some of the approaches, e.g. CFT and Laughlin's wavefunctions, are clearly very different and yet yield the same angles. Another way of looking at the agreement between the schemes is as a consistency check; maybe they are not all independent but at least there are no contradictions.

Looking at the angles themselves, we see that they all are of the form $\theta = r\pi/q$ where $\nu = p/q$ and $r \in \mathbb{N}$. The numerators are odd and less than π for self-statistics while they are even and larger than π for relative statistics. This points to a fundamental difference beween these two sorts. Where the clustering arguments only constrain the angle to within a set the smallest value is always the one realised.

With all models agreeing on the angles the likelihood that the results are correct is good. This is of course assuming that the statistics are robust, cf. section 7.1. Should an experiment measure a fractional QHE angle which deviates from those listed here, or is not anyonic at all, our understanding of the QHE has to be modified.

	Hierarchy	CF	CFT
ν	Q_E/e	Q_E/e	Q_E/e
1/m	-1/m	-1/m	-1/m
2/5	-1/5	-1/5	-1/5
2/9	-1/9	-1/9	-1/9
2/13	-1/13	-1/13	-1/13
2/17	-1/17	-1/17	-1/17
4/11	-1/11	-	-1/11
4/19	-1/19	-	-1/19
3/7	-1/7	-1/7	-1/7
3/13	-1/13	-1/13	-1/13
3/19	-1/19	-1/19	-1/19
7/17	-1/17	-	-1/17
7/19	-1/19	-	-1/19
4/9	-1/9	-1/9	-1/9
4/17	-1/17	-1/17	-1/17
5/11	-1/11	-1/11	-1/11
6/13	-1/13	-1/13	-1/13

Table 7.3: Electric charge of elementary qe's

	Hierarchy	CF	CFT	Clustering
ν	θ_S/π	θ_S/π	θ_S/π	θ_S/π
1/m	1/m	1/m	1/m	1/m
2/5	3/5	3/5	3/5	3/5
2/9	5/9	5/9	5/9	5/9
2/13	7/13	7/13	7/13	7/13
2/17	9/17	9/17	9/17	9/17
4/11	$\theta_2 : 3/11$	-	$\theta_1 : 5/11 \;\; \theta_2 : 3/11$	$\theta_1 : 5/11 \;\; \theta_2 : 3/11$
4/19	$\theta_2 : 5/19$	-	$\theta_1 : 7/19 \;\; \theta_2 : 5/19$	$\theta_1 : 7/19 \;\; \theta_2 : 5/19$
3/7	5/7	5/7	5/7	5/7
3/13	9/13	9/13	9/13	9/13
3/19	13/19	13/19	13/19	13/19
7/17	$\theta_3 : 5/17$	-	$\theta_1, \theta_2 : 11/17 \;\; \theta_3 : 5/17$	$\theta_1, \theta_2 : 11/17 \;\; \theta_3 : 5/17$
7/19	$\theta_3 : 11/19$	-	$\theta_1 : 9/19 \;\; \theta_2, \theta_3 : 11/19$	$\theta_1 : 9/19 \;\; \theta_2, \theta_3 : \;\; 11/19$
4/9	7/9	7/9	7/9	7/9
4/17	13/17	13/17	13/17	13/17
5/11	9/11	9/11	9/11	9/11
6/13	11/13	11/13	11/13	11/13

Table 7.4: Self-statistics angles of elementary qh's

	CFT	Clustering
ν	θ_R/π	θ_R/π
2/5	-2/5	-2/5 mod π
2/9	-4/9	-4/9 mod π
2/13	-6/13	-6/13 mod π
2/17	-8/17	-8/17 mod π
4/11	-2/11	-2/11 mod π
4/19	-4/19	-2/11 mod π
3/7	-2/7	-2/7 mod $\pi/3$
3/13	-4/13	-4/13 mod $\pi/3$
3/19	-6/19	-6/19 mod $\pi/3$
7/17	$\theta_{12}: -6/17$ $\theta_{13}, \theta_{23}: -2/17$	$\theta_{12}: -6/17$ $\theta_{13}, \theta_{23}: -2/17$ mod $\pi/7$
7/19	$\theta_{12}, \theta_{13}: -2/19$ $\theta_{23}: -8/19$	$\theta_{12}, \theta_{13}: -2/19$ $\theta_{23}: -8/19$ mod $\pi/7$
4/9	-2/9	-2/9 mod $\pi/6$
4/17	-4/17	-4/17 mod $\pi/6$
5/11	-2/11	-2/11 mod $\pi/10$
6/13	-2/13	-2/13 mod $\pi/15$

Table 7.5: Relative statistics angles of elementary qh's

	Hierarchy	CF	CFT	Clustering
ν	θ/π	θ/π	θ/π	θ/π
1/m	1/m	1/m	1/m	1/m
2/5	3/5	3/5	3/5	3/5
2/9	5/9	5/9	5/9	5/9
2/13	7/13	7/13	7/13	7/13
2/17	9/17	9/17	9/17	9/17
4/11	3/11	-	3/11	3/11
4/19	5/19	-	5/19	5/19
3/7	5/7	5/7	5/7	5/7
3/13	9/13	9/13	9/13	9/13
3/19	13/19	13/19	13/19	13/19
7/17	5/17	-	5/17	5/17
7/19	11/19	-	11/19	11/19
4/9	7/9	7/9	7/9	7/9
4/17	13/17	13/17	13/17	13/11
5/11	9/11	9/11	9/11	9/11
6/13	11/13	11/13	11/13	11/13

Table 7.6: Statistics angles of elementary qe's

7.5 Conclusion

In the course of this thesis we have examined the quantum Hall effect with emphasis on its quasiparticles and their anyonic statistics. We started out with a description of the QHE phenomenon and some basic theory, including an explanation of the integer effect. Then we introduced anyons, showing that they can only exist in less than three dimensions. The following three chapters reviewed three of the prevailing models describing the QHE:

Laughlin's approach supplies trial wavefunctions for the simplest fractions and their quasiparticles. These are justified with general arguments, referring to the Landau theory of a charged particle in a magnetic field. Jain's composite fermion model presents a picture where each electron binds itself to an odd number of unit fluxes, creating correlation holes around them so they stay apart. Using a mean field approximation where the magnetic field is smeared out the assumption is then that at certain filling fractions, the Jain sequence, the composite particles will produce an IQHE in the reduced magnetic field.

Through an effective low-energy description of the QHE, a Chern-Simons theory, Conformal field theory is connected to the QHE in two ways: The world lines of anyons in the CS theory can be expressed as correlators in a (2+0)-dimensional CFT, and the CS theory's dynamic degrees of freedom on the edge can be implemented in a (1+1)-dimensional CFT. These two are in fact the same, and in the context of the QHE all particles in CFT are represented as vertex operators usually with explicit monodromy.

The last chapter is the main part of the thesis. Here we generalized some arguments introduced by Su before summarizing the results and comparing the angles. We observed that there seems to be a close connection between charge and statistics, and also that relative statistics are in some ways distinct from the usual comcept of exchange statistics.

While the aim of this text is primarily to compare different models there are some new contributions as well. Mainly these are contained in the section concerning clustering arguments. Here we showed that Su's methods can be extended to quasielectrons and describinng several unequal quasiholes. We also gave an explicit proof of the one-to-one correspondence between charge and statistics and found restrictions on the relative as well as the self-statistics angles. The equations giving statistics angles from CFT, iterative such outside the Jain series, are also new.

The upshot from the comparison is that the models agree on the values of all the angles, where they can be compared. This is a strong indication that the results are right, whether one sees all the models as independent or if one does not but interpret the agreement as a consistency check. With experimental results we will see if this conclusion holds.

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